Mathematics

First form secondary First term

Egyptian Knowledge Bank
بنك المعرفة المصري

Mathematics has Practical applications in various fields including road construction, bridges and urban planning and preparing their maps which depend on parallel lines and their transversals according to the proportion between the real length and the drawing length.

Elsalam bridge connecting between the two shores of the Suez canel

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Introduction

بسم الله الرحمن الرحيم

we are pleased we offer this book to make it clear philosophy that has been in light construction of educational material and can be summarized as follows:

1. To emphasise that the main purpose of these books is to help the learner to solve problems and make decisions in their daily lives, and help them to participate in society.

2. Emphasis on the principle of continuity of life-long learning through work that students gain a systematic scientific thinking, and practice learning mixed with fun and suspense, relying on the development of problem- solving skills and develop the skills of the conclusion and reasoning, and the use of methods of self-learning, active learning and collaborative learning team spirit, and discussion and dialogue, and accept the opinions of others, and objectivity in sentencing, in addition to some definition of national activities and accomplishments.

3. Provide a comprehensive coherent visions of the relationship between science, technology and society (STS) reflect the role scientific progress in the development of the local community, in addition to focusing on the practice of conscious students to act effectively about to use technological instruments.

4. The development of positive attitudes towards the study of mathematics and aspect of its scientists.

5. To provide students with a comprehensive culture to use the available environmental resource.

6. Rely on the fundamentals of knowledge and develop methods of thinking, the development of scientific skills and stay away of the details and educational memorization, that's concern directed to bring concepts and general principles and research methods, problem solving and methods of thinking about the fundamental distinction mathematics from the others.

We have been especially cautions in this book the following:

★ The book has been divided into integrated and coherent units, for each one there is an introduction shows its aims, lessons, a short, and key terms, it has been divided into lessons explain the goal of study under the title"you will learn", each lesson starts with the main idea to the content of the lesson .It takes onto consideration, the presentation to the scientific article from easy to difficult and includes a set of activities that integrated with other subjects and to suit different abilities of students and take into consideration the individual differences between them and emphasizes the collaborative work, and integrated with the subject.

★ Every lesson has been presented examples from easy to difficult, it include variety of levels of thinking with drills on it under the title of " try to solve" and the lesson ends with a title of "check your understanding"

Eventually, we hope getting the right track for the benefits of our students as well as for our dearest Egypt hopping bright future to our dearest students. And the God of the intent behind, with leads to either way.

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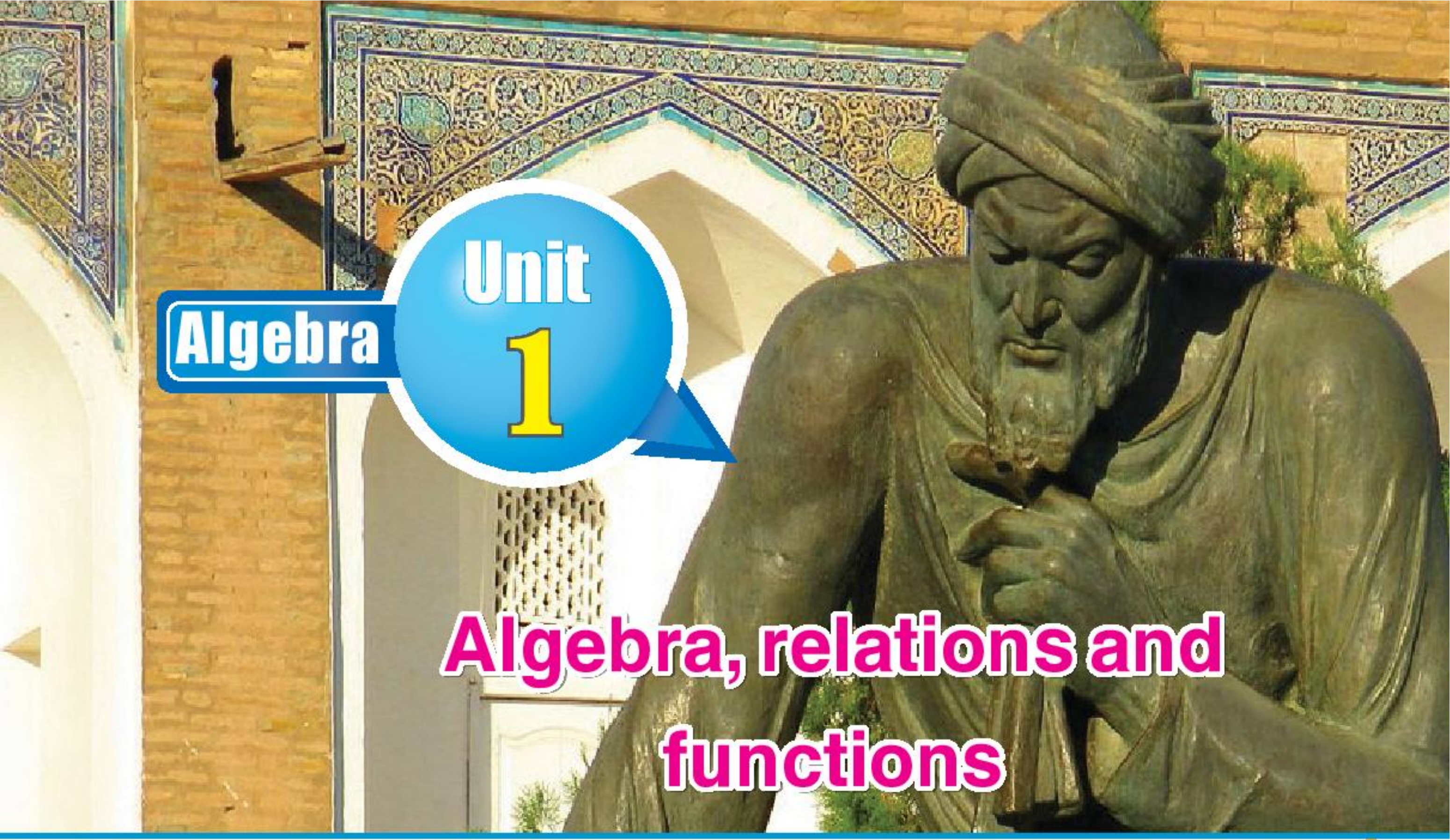


Algebra Unit 1

Algebra, relations and functions

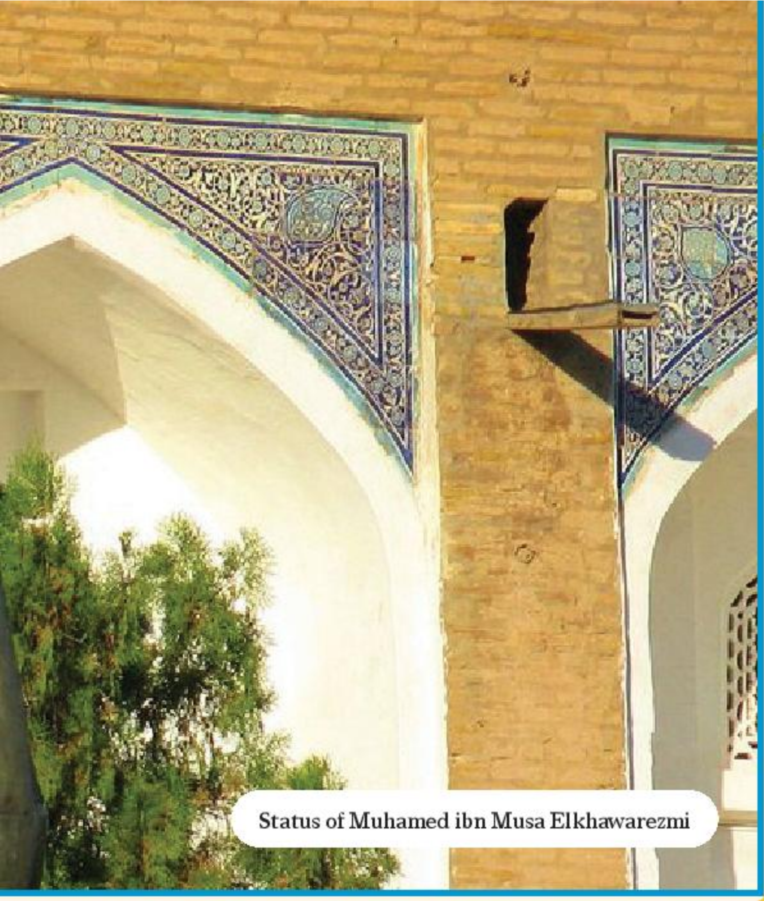
Unit objectives

By the end of the unit, the student should be able to:

- Find the sum and the product of the two roots of the quadratic equation in one variable .
- Find some of the coefficients of terms of the quadratic equation in terms of one of the two roots or both of them.
- Identify the discriminant of the quadratic equation in one variable.
- Investigate the type of the two roots of the quadratic equation in one variable in terms of the coefficients of its terms.
- Form the quadratic equation in one variable in terms of another quadratic equation in one variable.
- Investigate the sign of a function .
- Identify the introduction of the complex numbers (Definition of the complex number, powers of i, writing the complex number in the algebraic form, equality of two complex numbers)
- Solve quadratic inequalities in one variable.

Key - Terms

- Equation
- Root of the Equation
- Coefficient of a Term
- Discriminant of the Equation
- Sign of a function
- Complex Number
- Imaginary Number
- Powers of a Number
- Inequality



Status of Muhamed ibn Musa Elkhawarezmi

Lesson of the unit

Lesson (1 - 1): An Introduction in Complex Numbers.

Lesson (1 - 2): Determining the Types of the two Roots of a Quadratic Equation.

Lesson (1 - 3): The Relation Between the Two Roots of the Second Degree Equation and the Coefficients of its Terms.

Lesson (1 - 4): Sign of a Function.

Lesson (1 - 5): Quadratic Inequalities in one unknown

Materials

Scientific Calculator - Squared paper - Computer - Graphic Program - Electronic sites such as: www.phschool.com

Brief History

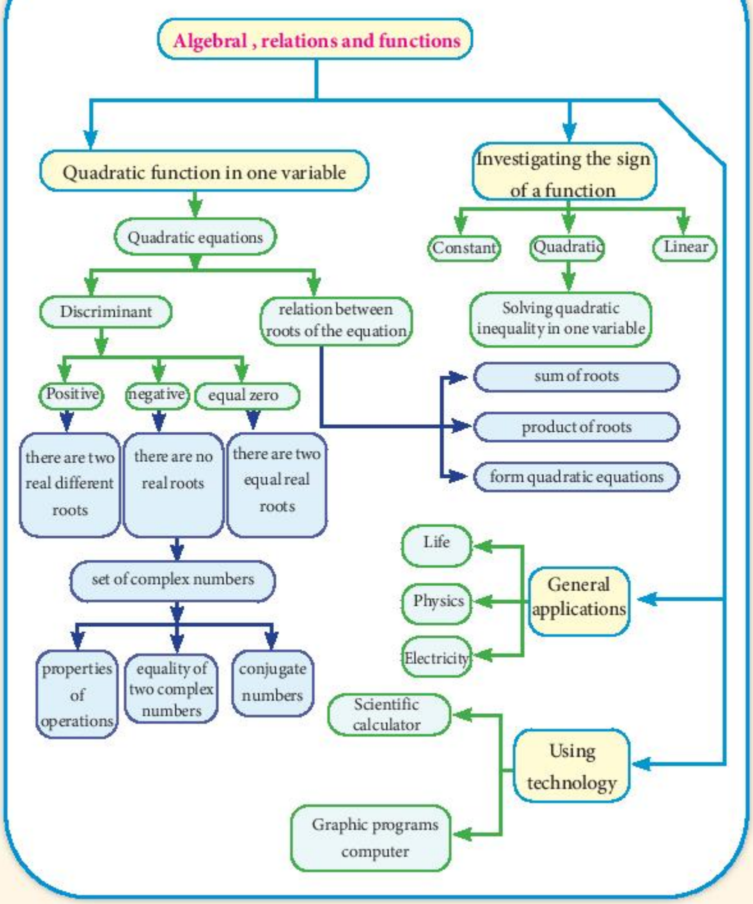
Algebra is an Arabic word used by Muhamed ibn Musa Elkhawarezmi (9th century AD, in the era of the Abbasy kalipha Elmamoon) . In his book (Algebra and AlMokabla) which contains the genuine ways to solve equation, So Elkawarezmi is the founder of algebra after it was a part of the calculation.

The book has been translated into European languages titled " Algebra" and taken from it the word "Algebra" And the root which we symbolized it currently by "x" (a reference to solve a quadratic equation). Elkawarizmi put Geometric solutions to the quadratic equation which match with the way of completing the square. Many Arabic scientists worked with solving equations, and the best known is omar Elkayyam who concerned with solving third degree equations.

It is worth mentioning that, it appeared in Ahmose papyrus (1860 BC), some of the problem that solutions refer that Egyptians at that time have found away to find the sum of the terms of an arithmetic and geometric sequences.

Now , Algebra is a large degree of sophistication and abstraction. it was dealing with numbers but now it has to deal with new mathematical entities such as : sets , matrices, vectors and so on, It is hopped you - our students - to restore the scientific glory in Golden eras through Egypt pharaonic and Islamic eras, Which our scientists Carry the banner of progress and flares knowledge to the world sides.

Chart of the unit



1 - 1 An Introduction in Complex Numbers

You will Learn

- Concept of the imaginary number.
- Integer power of i .
- Concept of the complex number.
- Equality of two complex numbers.
- Operations on the complex numbers.

Key - Terms

- Imaginary Number
- Complex Number

Materials

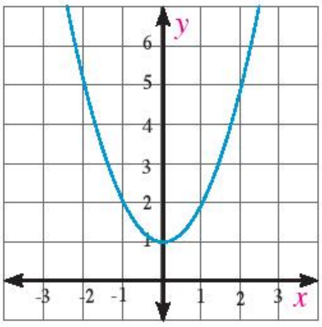
- Scientific calculator

Think and discuss

We have studied different systems of numbers which are : set of natural numbers N, set of integers Z , set of rational numbers Q, and set of irrational numbers Q', finally the set of real numbers $\mathbb{R}$. Any set of numbers is an extention to the set of numbers preceding it to solve new equations which have no solution in the preceding system, Look at the equation:

$x^2 = -1$, it is impossible to solve it in $\mathbb{R}$, because there is no real number whose square equals "-1" satsifies the equation, thus we need to study a new set of numbers which is called the set of complex numbers.

The figure opposite shows: the graph of the function $y = x^2 + 1$, from the graph we notice that the perabola does not intersect the x - axis. there is no real solution of the equation $x^2 + 1 = 0$ in $\mathbb{R}$ this is necessary to think about a new set of numbers to solve this kind of equations.



Learn

Imaginary numbers

The imaginary number "i" is defined as the number whose square equals (-1) **i . e** $i^2 = -1$ and $\sqrt{-n} = \sqrt{-1 \times n} = \sqrt{n}\ i$, when $n \in R^+$

and the numbers in the form 2i, - 5i, $\sqrt{3}$ i **are called the imaginary numbers**

So , we write $\sqrt{-3} = \sqrt{3}\ i$

$\sqrt{-5} = \sqrt{5}\ i$, ... and so on

Critical thinking: If a, and b are two negative real numbers. Is it possible that $\sqrt{a}\sqrt{b} = \sqrt{ab}$? explain your answer by giving a numerical example.

Integer powers of i

The number i satisfies the laws of indices which we studied before , it is possible to express the different powers of the number i as follows:

$i^1 = i$ $\quad$ $i^2 = -1$ $\quad$ $i^3 = i^2 \times i = -i$

$i^4 = i^2 \times i^2 = -1 \times -1 = 1$ $\quad$ $i^5 = i^4 \times i = 1 \times i = i$

In general : $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$ where $n \in Z$

Example

1 Find each of the following in the simplest form:

A i^{30} **B** i^{43} **C** i^{-61} **D** i^{4n+19}

Solution

A $i^{30} = (i^4)^7 \times i^2 = 1 \times -1 = -1$

B $i^{43} = (i^4)^{10} \times i^3 = 1 \times -i = -i$

C $i^{-61} = (i^4)^{-16} \times i^3 = 1 \times i^3 = -i$

D $i^{4n+19} = i^{4n} \times i^{19} = 1 \times (i^4)^4 \times i^3 = 1 \times i^3 = -i$

Try to solve

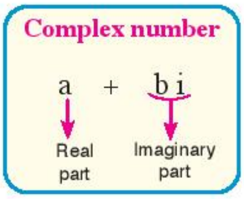
1 Find each of the following in the simplest form:

A i^{24} **B** i^{37} **C** i^{-43} **D** i^{-51} **E** i^{4n+29} **F** i^{4n+42}

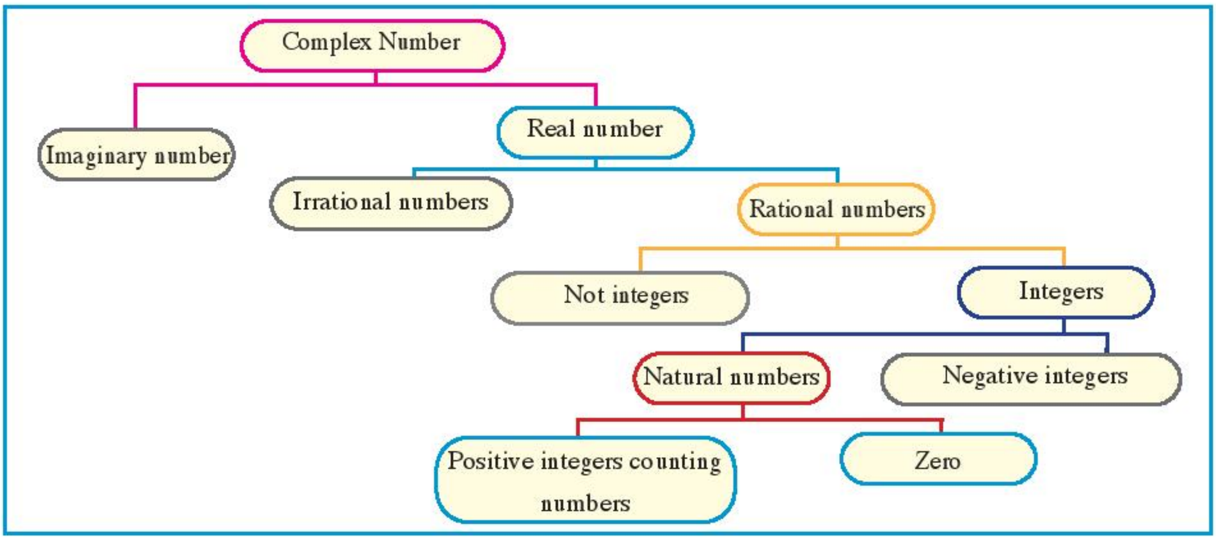
Learn

Complex number

The complex number is the number which can be written in the form $a + bi$ where a and b are real numbers.



The following figure shows the set of numbers which form a part of the system of the complex number.



If a and b are two real numbers, then the number z where z = a + b i, is called a complex number, where a is called the real part of the complex numer z and b i is the imaginary part of the complex number z.

If b = 0, then the number z = a is a pure real and if a = 0, then the number z = bi is a pure imaginary where $b \neq 0$

Example

2 Solve the equation $9x^2 + 125 = 61$

Solution

The equation $9x^2 + 125 = 61$

$9x^2 + 125 - 125 = 61 - 125$ **add (- 125) to both sides of the equation**

$9x^2 = -64$ **simplify**

$x^2 = -\frac{64}{9}$ **Divide both sides by 9**

$x = \pm\sqrt{\frac{-64}{9}}$ **by taking the square root of both sides**

$x = \pm\sqrt{\frac{64}{9}}\, i = \pm\frac{8}{3}\, i$ **Definition**

Try to solve

2 Solve each of the following equations:

A $3x^2 + 27 = 0$ B $5x^2 + 245 = 0$ C $4x^2 + 100 = 75$

Equality of two complex numbers

Two complex numbers are equal if and only if the two real parts are equal and the two imaginary parts are equal.

if: a + b i = c + d i **then**: a = c and b = d and vice versa.

Example

3 Find the values of x and y which satisfy the equation: $2x - y + (x - 2y)i = 5 + i$ where x and $y \in \mathbb{R}$ and $i^2 = -1$

Solution

By equalizing the two real parts and the two imaginary parts then

$2x - y = 5 \quad , \quad x - 2y = 1$

Solve the equations we get $x = 3 \quad , \quad y = 1$

Try to solve

3 Find the values of x and y which satisfy each of the following equations:

A $(2x + 1) + 4y\, i = 5 - 12\, i$

B $2x - 3 + (3y + 1)\, i = 7 + 10\, i$

Learn

Operations on complex numbers

It is possible to use the commutative, associative, and distributive properties to add or multiply complex numbers as shown in the following examples:

Example

(4) Find in the simplest form the result of each of the following:

(A) $(7 - 4i) + (2 + i)$

(B) $(2 + 3i)(3 - 4i)$

Solution

(A) **The expression** $(7 - 4i) + (2 + i)$

$= (7 + 2) + (-4 + 1)\,i$ **Commutative and associative properties**

$= 9 - 3\,i$ **Simplify**

(B) The expression $(2 + 3i)(3 - 4i)$

$= 2(3 - 4i) + 3i(3 - 4i)$ **Distributive property**

$= 6 - 8i + 9i - 12i^2$ **remove the parenthesis**

$= 6 - 8i + 9i + 12$ **where $i^2 = -1$**

$= (6 + 12) + (-8 + 9)\,i = 18 + i$ **Simplify**

Try to solve

(4) Find in the simplest form the result of each of the following:

(A) $(12 - 5i) - (7 - 9i)$

(B) $(4 - 3i)(4 + 3i)$

(C) $(5 - 6i)(3 + 2i)$

Conjugate Numbers

The two numbers $a + b\,i$ and $a - b\,i$ are called conjugate numbers

For example $4 - 3i$ and $4 + 3i$ are two conjugate numbers, where:

1) $(4 - 3i) + (4 + 3i) = (4)^2 - (3i)^2$ (the result as a real number)

$= 16 - 9i^2 = 16 - 9(-1) = 25$ **(The result is a real number)**

2) $(4 - 3i) + (4 + 3i) = 8$ (the result a real number)

Critical thinking:

Is the sum of two conjugate numbers always a real number? explain.

Is the product of two conjugate numbers always a real number? explain

Example

(5) Find the values of x and y which satisfy the equation:

$$\frac{(2+i)(2-i)}{3+4i} = x + i\,y$$

Solution

$\frac{4-i^2}{3+4i} = x + iy$	By removing the parentheses
$\frac{4+1}{3+4i} \times \frac{3-4i}{3-4i} = x + iy$	multiply up and down the L.H.S by 3 - 4i
$\frac{5(3-4i)}{25} = x + iy$	simplify
$\frac{3}{5} - \frac{4}{5}i = x + yi$	apply the equality of two complex numbers

i.e.: $x = \frac{3}{5}$, $y = -\frac{4}{5}$

Try to solve

(5) Find in the simplest form , the value of each of the following:

(A) $\frac{4-6i}{2i}$ (B) $\frac{26}{3-2i}$

(C) $\frac{3-i}{2-i}$ (D) $\frac{3+4i}{5-2i}$

Example

(6) **Electricity:** Find the total electric current intensity passing through two resistances connected on parallel in a closed circuit, if the current intensity in the first resistance is $5 - 3i$ ampere and the second resistance is $2 + i$ ampere (given that the total current intensity equals the sum of the two current intensities which passes the two resistances)

Solution

$\because$ The total electric current intensity = the sum of the two current intensities passing in the two resistances.

$\therefore$ $= (5 - 3i) + (2 + i)$

$= (5 + 2) + (-3 + 1)\,i$

$= 7 - 2i$ ampere

Check your understanding

1) **Critical thinking:** Find in the simplest form $(1- i)^{10}$.

Exercises (1 - 1)

1) Simplify:

A) i^{66} B) i^{-45} C) i^{4n+2} D) i^{4n-1}

2) Simplify:

A) $\sqrt{-18} \times \sqrt{-12}$ B) $3i(-2i)$ C) $(-4i)(-6i)$ D) $(-2i)^3(-3i)^2$

3) Find in the simplest form:

A) $(3 + 2i) + (2 - 5i)$ B) $(26 - 4i) - (9 - 20i)$

C) $(20 + 25i) - (9 - 20i)$

4) Rewrite each of the following in the form $a + bi$

A) $(2 + 3i) - (1 - 2i)$ B) $(1 + 2i^3)(2 + 3i^5 + 4i^6)$

5) Rewrite each of the following in the form $a + bi$

A) $\frac{2}{1+i}$ B) $\frac{4+i}{i}$ C) $\frac{2-3i}{3+i}$ D) $\frac{(3+i)(3-i)}{3-4i}$

6) Solve each of the following equations:

A) $3x^2 + 12 = 0$ B) $4y^2 + 20 = 0$ C) $4z^2 + 72 = 0$ D) $\frac{3}{5}y^2 + 15 = 0$

7) **Discover the error:** Find the simplest form of the expression: $(2 + 3i)^2(2 - 3i)$

Ahmed's answer

$(2 + 3i)(2 + 3i)(2 - 3i)$
$= (2 + 3i)(4 - 9i^2)$
$= (2 + 3i)(4 + 9) = 13(2 + 3i)$
$= 26 + 39i$

Karim's answer

$(2 + 3i)^2(2 - 3i) = (4 + 9i^2)(2 - 3i)$
$= (4 - 9)(2 - 3i) = -5(2 - 3i)$
$= -10 + 15i$

Which solutions is correct ? Why?

1 - 2 Determining the Types of Roots of a Quadratic Equation

You will Learn

- How to determine the type of the two roots of the quadratic equation

Key - Terms

- Root
- Discriminant

Materials

- Scientific Calculator

Think and discuss

You have previously studied how to solve the quadratic equation in one variable in $\mathbb{R}$, and you have known that a quadratic equation has two roots or a unique repeated solution, or there is no solution. Is it possible to find the number of roots (solutions) of the quadratic equation in $\mathbb{R}$ without solving it?

Learn

Discriminant

The two roots of the quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$ and $a, b, c \in \mathbb{R}$

are: $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$, $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$

and the two roots contain the expression $\sqrt{b^2 - 4ac}$.

the expression $b^2 - 4ac$ is called the discriminant of the quadratic equation and is used to determine the type of the two roots of the equation.

Example

1 **Determine the type of the two roots of each of the following equations:**

A $5x^2 + x - 7 = 0$ **B** $x^2 - 2x + 1 = 0$

C $-x^2 + 5x - 30 = 0$

Solution

To determine the type of the two roots:

A $a = 5$, $b = 1$, $c = -7$

the discriminant $= b^2 - 4ac$

$= 1 - 4 \times 5 (-7) = 141$

$\because$ the discriminant is positive, $\therefore$ there are two real different roots.

B $a = 1$, $b = -2$, $c = 1$

the discriminant $= b^2 - 4ac$

$= 4 - 4 \times 1 \times 1 = 0$

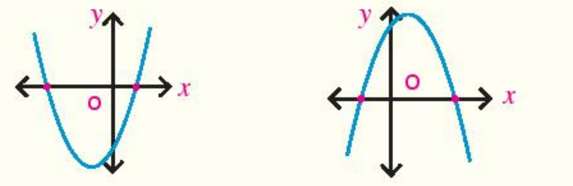
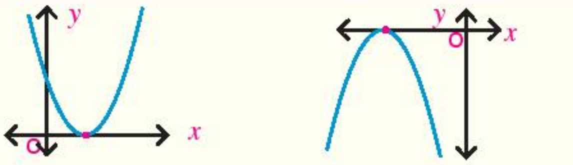
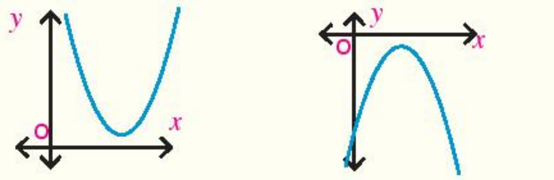
$\because$ the discriminant equals zero, $\therefore$ the the two roots are real and equal.

(C) a = -1 , b = 5 , c = - 30 the discriminant = $b^2 - 4ac$

$= 25 - 4 \times -1 \times -30 = -95$

$\because$ the discriminant is negative, $\therefore$ there are two complex non real and conjugate roots.

Notice that

Discriminant	The type of the two roots	Sketch of the function related to the equation
$(b^2 - 4ac) > 0$	**two real different roots**	
$b^2 - 4ac = 0$	**a real repeated root two equal roots**	
$b^2 - 4ac < 0$	**two complex and non real roots (Conjugate)**	

Try to solve

(1) Determine the type of the two roots for each of the following equations :

(A) $6x^2 = 19x - 15$ (B) $12x - 4x^2 = 9$

(C) $x(x - 2) = 5$ (D) $x(x + 5) = 2(x - 7)$

Example

(2) Prove that the two roots of the equation $2x^2 - 3x + 2 = 0$ are complex and not real, then use the general formula to find two roots roots.

Solution

a = 2, b = - 3 and c = 2

$\because$ the discriminant $= b^2 - 4ac$ $\therefore$ the discriminant $= (-3)^2 - 4 \times 2 \times 2 = 9 - 16 = -7$

$\because$ **the discriminant is negative** $\therefore$ **there are two complex roots and not real**

The general formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-3) \pm \sqrt{7i^2}}{2 \times 2} = \frac{3 \pm \sqrt{7}\, i}{4}$$

The two roots of the equation are: $\frac{3}{4} + \frac{\sqrt{7}}{4} i$ and $\frac{3}{4} - \frac{\sqrt{7}}{4} i$

Critical thinking: If the discriminate is negative, is it necessary that the two roots of the quadratic equation in the set of complex numbers are conjugate numbers? Give an example to explain

Try to solve

2) Prove that the two roots of the equation $7x^2 - 11x + 5 = 0$ are complex, then use the general formula to find those two roots.

Example

3) If the two roots of the equation $x^2 + 2(k - 1)x + 9 = 0$ are equal, then find the real values of K and check your answer:

Solution

$b^2 - 4ac = 0$
$4(k - 1)^2 - 4 \times 1 \times 9 = 0$
$4k^2 - 8k - 32 = 0$
$k^2 - 2k - 8 = 0$
$(k - 4)(k + 2) = 0$
$k = 4$ or $k = -2$

check: when $k = 4$
the equation becomes: $x^2 + 6x + 9 = 0$
and it has two equal roots which are: -3, -3
check when $k = -2$
the equation becomes: $x^2 - 6x + 9 = 0$
and it has two equal roots which are: 3,3

Try to solve

3) If the two roots of the equation $x^2 - 2kx + 7k - 6x + 9 = 0$ are equal , then find the real values of K and then find the two roots.

Exercises (1 - 2)

First: Multiple choice:

1) The two roots of the equation $x^2 - 4x + k = 0$ are equal if:

(A) k = 1 (B) k = 4 (C) k = 8 (D) k = 16

2) The two roots of the equation $x^2 - 2x + M = 0$ are real different if :

(A) M = 1 (B) M > 1 (C) M > 1 (D) M = 4

3) The two roots of the equation $Lx^2 - 12x + 9 = 0$ are complex and not real if :

(A) L < 4 (B) L > 4 (C) L = 4 (D) L = 1

Second: Answer the following questions:

4) Determine the number of roots and their types in the following quadratic equation:

(A) $x^2 - 2x + 5 = 0$ (B) $3x^2 + 10x - 4 = 0$

(C) $x^2 - 10x + 25 = 0$ (D) $6x^2 - 19x + 35 = 0$

5) Find the solution of the following equations in the set of complex numbers using the general formula.

A) $x^2 - 4x + 5 = 0$

B) $2x^2 + 6x + 5 = 0$

C) $3x^2 - 7x + 6 = 0$

D) $4x^2 - x + 1 = 0$

6) Find the value of K in each of the following cases:

A) If the two roots of the equation $x^2 + 4x + K = 0$ are real different.

B) If the two roots of the equation $x^2 - 3x + 2 + \frac{1}{K} = 0$ are equal.

C) If the two roots of the equation $Kx^2 - 8x + 16 = 0$ are complex and not real.

7) **Discover the error:** What is the number of solutions of the equation $2x^2 - 6x = 5$ in R

Ahmed's answer

$b^2 - 4ac = (-6)^2 - 4 \times 2 \times 5$

$= 36 - 40 = -4$

The discriminant is negative, then there is no real solutions

Karim's answer

$b^2 - 4ac = (-6)^2 - 4 \times 2\,(-5)$

$= 36 + 40 = 76$

the discriminant is positive, then there are two real different solutions

8) If the two roots of the equation $x^2 + 2(K - 1)x + (2K + 1) = 0$ are equal, then find the real values of K, and the two roots.

9) **Critical thinking:** solve the equation $36x^2 - 48x + 25 = 0$ in the set of complex numbers.

1 - 3 Relation Between the Two Roots of the Second Degree Equation and the Coefficients of its Terms

You will Learn

- How to find the sum of the two roots of a given quadratic equation.
- How to find the product of the two roots.
- Finding a quadratic equation in terms of another quadratic equation

Key - Terms

- Sum of Two Roots
- Product of Two Roots

Materials

- Scientific calculator

Think and discuss

We know that the two roots of the equation $4x^2 - 8x + 3 = 0$ are $\frac{1}{2}$ and $\frac{3}{2}$.

Sum of the two roots $\frac{1}{2} + \frac{3}{2} = \frac{1+3}{2} = 2$

Product of the two roots $\frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$

Is there a relation between the sum of the two roots of the equation and coefficients of its terms?

Is there a relation between the product of the two roots of the equation and coefficients of its terms?

Learn

Sum and product of two roots

The two roots of the quadratic equation $ax^2 + bx + c = 0$ are:

$\frac{-b + \sqrt{b^2 - 4ac}}{2a}$, $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

let the first root be L and the other root be M then :

$L + M = \frac{-b}{a}$ (Prove that) $\qquad L M = \frac{c}{a}$ (Prove that)

Oral exercise In the quadratic equation $ax^2 + bx + c = 0$

find $L + M$ and $L M$ in each of the following cases:

A If $a = 1$ **B** If $b = a$ **C** If $a = c$

Example

1 Without solving the equation, find the sum and the product of the two roots of the equation: $2x^2 + 5x - 12 = 0$

Solution

$a = 2$, $b = 5$, $c = -12$

Sum of the two roots $= \frac{-b}{a} = \frac{-5}{2} = -\frac{5}{2}$

product of the two roots $= \frac{c}{a} = \frac{-12}{2} = -6$

Try to solve

1. Without solving the equation , find the sum and product of the two roots in each of the following equations :

A) $2x^2 + x - 6 = 0$ B) $3x^2 = 23x - 30$ C) $(2x - 3)(x + 2) = 0$

Example

2. If the product of the two roots of the equation $2x^2 - 3x + k = 0$ equals 1, find the value of k, then solve the equation.

Solution

product of the two roots $= \frac{c}{a}$ $\therefore \frac{k}{2} = 1$ $\therefore k = 2$

a = 2, b = - 3, c = 2

The general formula: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{3 \pm \sqrt{9-16}}{4} = \frac{3 \pm \sqrt{7i^2}}{4} = \frac{3 \pm \sqrt{7}\,i}{4}$

The solution set of the equation is $\{\frac{3}{4} + \frac{\sqrt{7}}{4} i , \frac{3}{4} - \frac{\sqrt{7}}{4} i\}$

Try to solve

2. If the product of the two roots of the equation $3x^2 + 10x - c = 0$ is $\frac{-8}{3}$, find the value of c, then solve the equation.
3. If the sum of the two roots of the equation $2x^2 + bx - 5 = 0$ is $-\frac{3}{2}$, find the value of b, then solve the equation.

Example

3. If $(1 + i)$ is one of the roots of the equation $x^2 - 2x + a = 0$ where $a \in R$, then find:

A) The other root B) the value of a .

Solution

a = 1 , b = -2 , c = a

A) $\because$ $1 + i$ is one of the two roots of the equation

$\therefore$ the other root $= 1 - i$ **because the two roots are conjugate and their sum = 2**

B) $\because$ Product of the roots $= a$

$\therefore (1 + i)(1 - i) = a$

$\therefore 1 + 1 = a$

$\therefore a = 2$

Try to solve

4. If $(2 + i)$ is one of the roots of the equation $x^2 - 4x + b = 0$. where $b \in R$ then find:
   A the other root. **B** the value of b.

Learn

Forming the quadratic equation whose two roots are known

Let L and M be the roots of the quadratic equation: $ax^2 + bx + c = 0, a \neq 0$

Dividing both sides of the equation by a: $\therefore x^2 + \frac{b}{a} x + \frac{b}{a} = 0$

i.e. $x^2 - (\frac{-b}{a}) x + \frac{c}{a} = 0$

$\because L$ and M are the two roots of the quadratic equation , $L + M = -\frac{b}{a}$, $LM = \frac{c}{a}$

$\therefore$ **The quadratic equation whose roots are L and M is:**

$$x^2 - (L + M)x + LM = 0$$

Example

4. Form the quadratic equation whose two roots are 4 and -3 .

Solution

Let the two roots of the equation be l and M

$\because L + M = 4 + (-3) = 1, LM = 4(-3) = -12,$

$\because$ **The quadratic equation formula is:** $x^2 - (L + M)x + LM = 0$

$\therefore$ **The equation is:** $x^2 - x - 12 = 0$

Example

5. Form the quadratic equation whose two roots are: $\frac{-2+2i}{1+i}$ and $\frac{-2-4i}{2-i}$

Solution

Let the two roots be L and M

$L = \frac{-2+2i}{1+i} \times \frac{1-i}{1-i} = \frac{4i}{2} = 2i$

$M = \frac{-2-4i}{2-i} \times \frac{2+i}{2+i} = \frac{-10i}{5} = -2i$

$L + M = 2i - 2i = 0$

, $LM = 2i \times -2i = -4i^2 = 4$

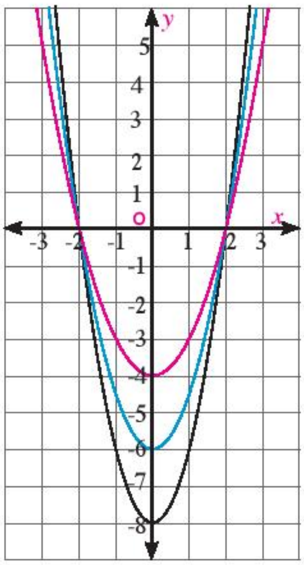
$\because$ The quadratic equation whose two roots are L and M is: $x^2 - (L + M)x + LM = 0$

$\therefore x^2 + 4 = 0$

Try to solve

5. Form the quadratic equation in each of the following given its two roots:
   A 3 , -5 **B** $-9i$, $9i$ **C** $\frac{3}{i}$, $\frac{3+3i}{1-i}$

Critical thinking: the figure opposite represents a set of parabolas of some quadratic function which each of them passes through the points (0, -2) , (0, 2).
Find the rule of each function



Forming a quadratic equation from the roots of another equation

Example

6 If L and M are the two roots of the equation $2x^2 - 3x - 1 = 0$, then form the quadratic equation whose roots are L^2 and M^2.

Solution

The given equation: by substituting a = 2, b = - 3, c = -1:

$L + M = -\frac{-3}{2} = \frac{3}{2}$, $L M = -\frac{1}{2}$

The required equation by substituting $L + M = \frac{3}{2}$, $L M = -\frac{1}{2}$ in the formula

$\therefore L^2 + M^2 = (L + M)^2 - 2LM = (\frac{3}{2})^2 - 2\ (-\frac{1}{2})$

$= \frac{9}{4} + 1 = \frac{9}{4} + \frac{4}{4} = \frac{13}{4}$

$\because L^2M^2 = (LM)^2$

$\therefore L^2M^2 = (-\frac{1}{2})^2 = \frac{1}{4}$

Notice that

$L^2 + M^2 = (L + M)^2 - 2LM$

$(L - M)^2 = (L + M)^2 - 4LM$

By substituting in the formula of the quadratic equation:

x^2 – (sum of the two roots) x + product of the two roots = 0

$x^2 - \frac{13}{4}x + \frac{1}{4} = 0$ **Multiply both sides of the equation by 4**

$\therefore$ **The required quadratic equation is:** $4x^2 - 13x + 4 = 0$

Try to solve

6 In the previous equation $2x^2 - 3x - 1 = 0$, form the quadratic equations whose each of its two roots are as follows:

A $\frac{1}{L}, \frac{1}{M}$ **B** $\frac{L}{M}, \frac{M}{L}$ **C** $L + M$, LM

Check your understanding

1 In each of the following, form the quadratic equation whose two roots are:

A $\frac{3}{4}, \frac{4}{3}$ **B** $5\sqrt{3}$, $-2\sqrt{3}$ **C** $3 + \sqrt{2}\,i$, $3 - \sqrt{2}\,i$

2 If L and M are the two roots of the equation $x^2 + 3x - 5 = 0$, then form the quadratic equation whose roots are L^2, M^2.

Exercises (1 - 3)

First: Complete each of the folowing:

1. if $x = 3$ is one of the roots of the equation $x^2 + Mx - 27 = 0$, then M = ... and the other root is ....
2. If the product of the two roots of the equation : $2x^2 + 7x + 3K = 0$ equals the sum of the two roots of the equation: $x^2 - (K + 4)x = 0$, then K = ...
3. The quadratic equation which each of its two roots increases 1 than each of the two roots of the quadratic equation $x^2 - 3x + 2 = 0$ is ....
4. The quadratic equation which each of its two roots decreases 1 than each of the two roots of the quadratic equation $x^2 - 5x + 6 = 0$ is ....

Second: multiple choice

5. If one of the two roots of the equation $x^2 - 3x + c = 0$ is twice the other, then c=
   (A) -4 (B) -2 (C) 2 (D) 4
6. If one of the two roots of the equation $ax^2 - 3x + 2 = 0$ is the multiplicative inverse of the other , then a =
   (A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) 2 (D) 3
7. If one of the two roots of the equation $x^2 - (b - 3)x + 5 = 0$ is the additive inverse of the other, then b =
   (A) - 5 (B) - 3 (C) 3 (D) 5

Third : Answer the following questions

8. Find the sum and the product of the two roots in each of the following equations:
   (A) $3x^2 + 19x - 14 = 0$ (B) $4x^2 + 4x - 35 = 0$
   ....
9. Find the value of a , then find the other root in each of the following equations:
   (A) If: $x = -1$ is one of the two roots of the equation $x^2 - 2x + a = 0$ ....
   (B) If: $x = 2$ is one of the two roots of the equation $ax^2 - 5x + a = 0$ ....
10. Find the values of a and b if:
    (A) 2 and 5 are the two roots of the equation $x^2 + ax + b = 0$
    (B) -3 and 7 are the two roots of the equation $ax^2 - bx - 21 = 0$
    (C) -1and $\frac{3}{2}$ are the two roots of the equation $ax^2 - x + b = 0$
    (D) $\sqrt{3}i$ and $-\sqrt{3}i$ are the two roots of the equation $x^2 + ax + b = 0$

11 Investigate the type of the two roots in each of the following equations, then find the solution set of each equation:

A) $x^2 + 2x - 35 = 0$ B) $2x^2 + 3x + 7 = 0$

C) $x(x - 4) + 5 = 0$ D) $3x(3x - 8) + 16 = 0$

12 Find the value of c, if the two roots of the equation $c\,x^2 - 12x + 9 = 0$ are equal.

13 Find the value of a, if the two roots of the equation $x^2 - 3x + 2 + \frac{1}{a} = 0$ are equal.

14 Find the value of c, if the two roots of the equation $3x^2 - 5x + c = 0$ are equal, then find the two roots.

15 Find the value of K, if one a root of the equation $x^2 + (K - 1)x - 3 = 0$ is the additive inverse to the other root.

16 Find the value of K, if one root of the equation $4Kx^2 + 7x + K^2 + 4 = 0$ is the multiplicative inverse to the other root.

17 Form the quadratic equation whose two roots are :

A) $-2, 4$ B) $-5i, 5i$ C) $\frac{2}{3}, \frac{3}{2}$ D) $1 - 3i, 1 + 3i$

E) $3 - 2\sqrt{2}i$, $3 + 2\sqrt{2}i$

18 Find the quadratic equation in which each of the two roots is twice one of the roots of the equation $2x^2 - 8x + 5 = 0$

19 Find the quadratic equation in which each of the two roots exceeds 1 than one of the two roots of the equation : $x^2 - 7x - 9 = 0$

20 Find the quadratic equation in which each of its two roots equals the square of the corresponding root of the equation : $x^2 + 3x - 5 = 0$

21 If L and M are the two roots of the equation $x^2 - 7x + 3 = 0$, then find the quadratic equation whose roots are:

A) $2L, 2M$ B) $L + 2, M + 2$ C) $\frac{2}{L}, \frac{2}{M}$ D) $L + M, LM$

1 - 4 Sign of a Function

You will Learn

- Investigate the sign of: constant function - Linear function - quadratic function.

Key - Terms

- Sign of a function
- Constant Function
- Linear Function
- Quadratic Function

Materials

- Scientific Calculator

Think and discuss

You have studied before the graphic representation of a linear and quadratic functions, and recognized the general figure of the curve of each function, can you investigate the sign of each of these functions? We mean by investigating the sign of the function to determine the values of the variable x (domain of x) at which the values of the function f are as follows:

Positive, means $f(x) > 0$

negative, means $f(x) < 0$

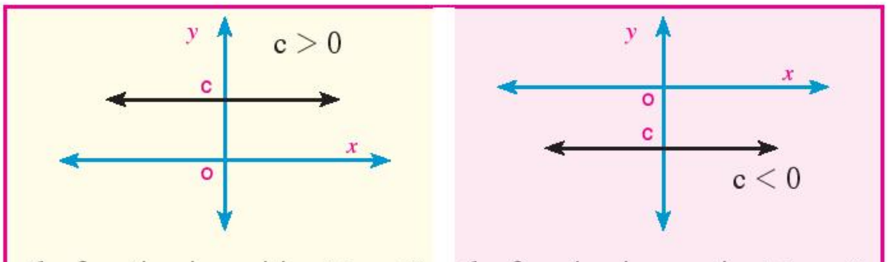
equal to zero, means $f(x) = 0$

Learn

First: Sign of the Constant Function

Sign of the constant function f where $f(x) = c$ $(c \neq 0)$ is the same as the sign of c $\forall x \in \mathbb{R}$.

and the following figure shows the sign of the function f.



the function is positive $\forall x \in \mathbb{R}$ | the function is negative $\forall x \in \mathbb{R}$

Example

1 Determine the sign of each of the following functions:

A $f(x) = 5$ **B** $f(x) = -7$

Solution

A $\because f(x) > 0$ $\therefore$ Sign of the function is positive $\forall x \in \mathbb{R}$

B $\because f(x) < 0$ $\therefore$ Sign of the function is negative $\forall x \in \mathbb{R}$

Try to solve

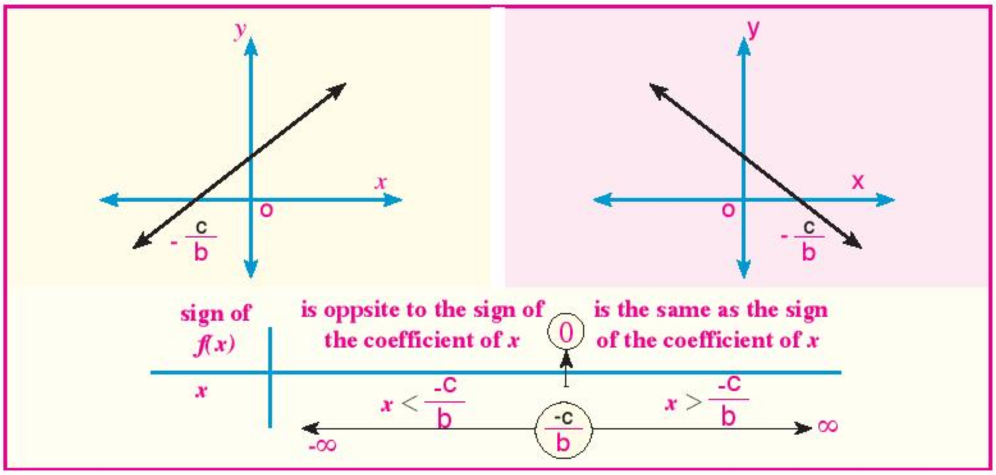
1) Determine the sign of each of the following functions:

(A) $f(x) = -\frac{2}{3}$ (B) $f(x) = \frac{5}{2}$

Second: Sign of the Linear Function

the rule of the function f is $f(x) = bx + c$, $b \neq 0$, $x = -\frac{c}{b}$ when $f(x) = 0$

and the following figure shows the sign of the function f.



Example

2) Determine the sign of the function f where $f(x) = x - 2$ and represent it graphically:

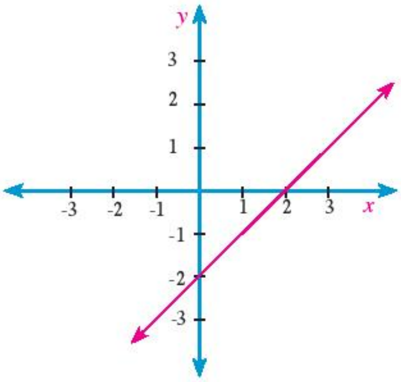
Solution

The rule of the function: $f(x) = x - 2$

Drawing the graph of the function:

when $f(x) = 0$ then $x = 2$

when $x = 0$ then $f(0) = -2$



From the graph, we get:

- The function is positive when $x > 2$
- The function $f(x) = 0$ when $x = 2$
- The function is negative when $x < 2$

Try to solve

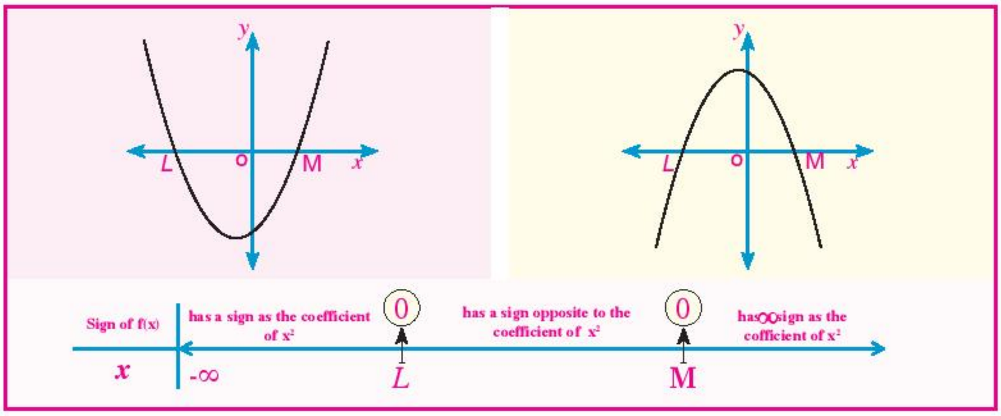
2) Determine the sign of the function $f(x) = -2x - 4$ and represent it graphically.

Third: Sign of the Quadratic Function.

To determine the sign of the quadratic function f, where $f(x) = ax^2 + bx + c$

We find the discriminant of the equation $ax^2 + bx + c = 0$ if:

First: $b^2 - 4ac > 0$ then there are two real roots L and M of the equation, and let $L <$ M, then the sign of the function is as in the following figure:



Example

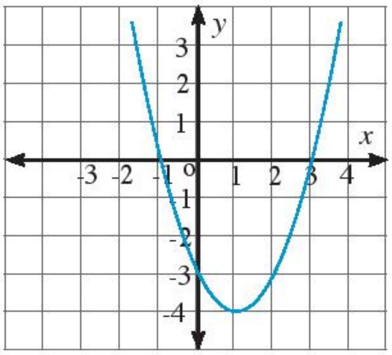
3 Represent graphically f, where $f(x) = x^2 - 2x - 3$, then determine the sign of the function f.

Solution

By factorizing the equation: $x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

Then the two roots of the equation are: -1, 3



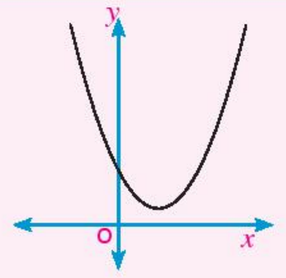
From the graph , we get:

- $f(x) > 0$ when $x \in \mathbb{R} - [-1, 3]$
- $f(x) < 0$ when $x \in]-1, 3[$
- $f(x) = 0$ when $x \in \{-1, 3\}$

Try to solve

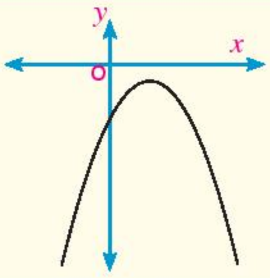
3 Represent graphically f, where $f(x) = x^2 - x + 6$, then determine the sign of the function f.

Second: If: $b^2 - 4ac < 0$, then there are no real roots and the sign of the function f is the same as the sign of the coefficient of x^2, and the following figures show that.



If: $a > 0$

$f(x) > 0 \ \forall\, x \in \mathbb{R}$



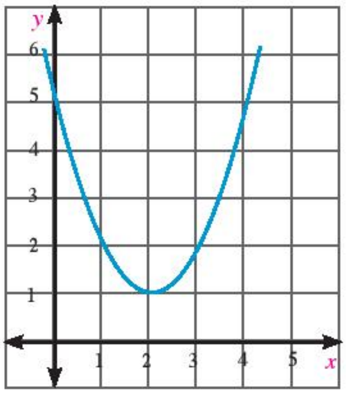
If: $a < 0$

$f(x) < 0 \ \forall\, x \in \mathbb{R}$

Example

4 Represent graphically f where $f(x) = x^2 - 4x + 5$, then determine the sign of the function f.

Solution



the discriminant $(b^2 - 4\,ac) = (-4)^2 - 4 \times 1 \times 5$

$= 16 - 20 = -4 < 0$

thus, the equation $x^2 - 4x + 5 = 0$ has no real roots

Sign of the function is positive $\forall\ x \in \mathbb{R}$

(because the coefficient of $x^2 > 0$)

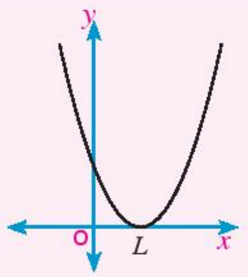
Try to solve

4 Represent graphically f, where $f(x) = -x^2 - 2x - 4$, then determine the sign of the function f.

Third: If: $b^2 - 4ac = 0$, then there are two equal roots of the equation, let each of them equal L, and the sign of the function f is as follows:

- the same as a when $x \neq L$
- $f(x) = 0$ when $x = L$

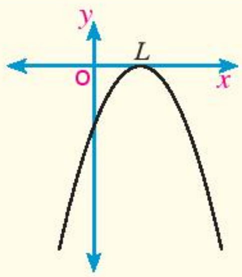
the following figures show that.



If: $a > 0$

$f(x) > 0 \ \forall\ x \neq L,$

$f(x) = 0$ when $x = L$



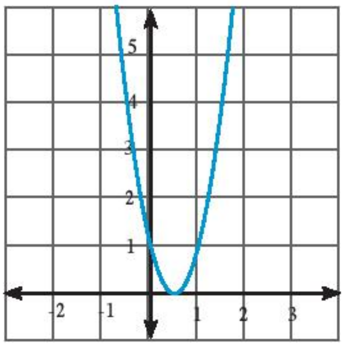
If: $a < 0$

$f(x) < 0 \ \forall\ x \neq L,$

$f(x) = 0$ when $x = L$

Example

(5) Represent graphically f where $f(x) = 4x^2 - 4x + 1$, then determine the sign of the function f.



Solution

The discriminant $(b^2 - 4ac) = (-4)^2 - 4 \times 4 \times 1$

$= 16 - 16 = 0$

Thus, the equation $4x^2 - 4x + 1 = 0$ has two equal roots.

By factorization: $(2x - 1)^2 = 0$

Put: $2x - 1 = 0$ then $x = \frac{1}{2}$

$f(x) > 0$ when $x \neq \frac{1}{2}$, $f(x) = 0$ when $x = \frac{1}{2}$

Try to solve

(5) Represent graphically f, where $f(x) = -4x^2 - 12x - 9$, then determine the sign of the function f.

Example

(6) Prove that: for all values of $x \in R$, the two roots of the equation $2x^2 - kx + k - 3 = 0$ are real different.

Solution

The discriminant $(b^2 - 4ac) = (-k)^2 - 4x\ 2x\ (k - 3) = k^2 - 8k + 24$

The two roots of the equation are real different if the discriminant is positive.

Investigate the sign of the expression $y = k^2 - 8k + 24$

The discriminant of the equation $k^2 - 8k + 24 = 0$ **is:**

$(-8)^2 - 4 \times 1 \times 24 = 64 - 96 = -32 < 0$

Thus the equation	$k^2 - 8k + 24 = 0$	has no real roots
∴ **sign of the expression**	$y = k^2 - 8k + 24$	is positive $\forall k \in \mathbb{R}$ (why)?
Then the discriminant of:	$2x^2 - kx + k - 3 = 0$	is positive $\forall x \in \mathbb{R}$
∴ **The two roots of the equation**	$2x^2 - kx + k - 3 = 0$	are real different $\forall x \in \mathbb{R}$

Check your understanding

(1) Determine the sign of each of the following functions:

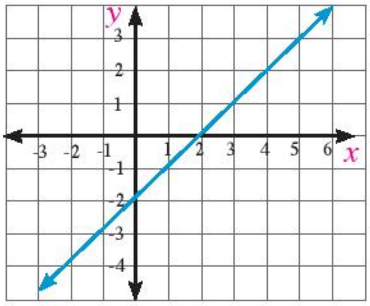
(A) $f(x) = 2x - 3$ (B) $f(x) = 4 - x$ (C) $f(x) = x^2 - 4$

(D) $f(x) = 1 - x^2$ (E) $f(x) = 4 + 4x + x^2$ (F) $f(x) = 3x - 2x^2 + 4$

Exercises (1 - 4)

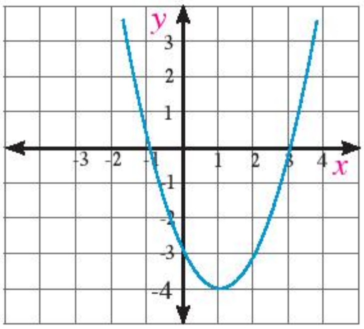
First : complete each of the following:

1. The sign of the function f, where $f(x) = -5$ is .... in the interval ....
2. The sign of the function f, where $f(x) = x^2 + 1$ is .... in the interval ....
3. The sign of the function f, where $f(x) = x^2 - 6x + 9$ is positive in the interval ....
4. The sign of the function f, where $f(x) = x - 2$ is positive in the interval ....
5. The sign of the function f, where $f(x) = 3 - x$ is negative in the interval ....
6. The sign of the function f, where $f(x) = -(x - 1)(x + 2)$ is positive in the interval ....
7. The sign of the function f, where $f(x) = x^2 + 4x - 5$ is negative in the interval ....
8. The figure opposite represents a first degree function in x:



   A. The function is positive in the interval ....
   B. the function is negative in the interval ....

9. The figure opposite represents a second degree function in x:



   A. $f(x) = 0$ when $x \in$ ....
   B. $f(x) < 0$ when $x \in$ ....
   C. $f(x) > 0$ when $x \in$ ....

Second: answer the following questions:

10) In exercises from (A) to (N), determine the sign of each of the following functions:

(A) $f(x) = 2$ ....

(B) $f(x) = 2x$ ....

(C) $f(x) = -3x$ ....

(D) $f(x) = 2x + 4$ ....

(E) $f(x) = 3 - 2x$ ....

(F) $f(x) = x^2$ ....

(G) $f(x) = 2x^2$ ....

(H) $f(x) = x^2 - 4$ ....

(I) $f(x) = 1 - x^2$ ....

(J) $f(x) = (x - 2)(x + 3)$ ....

(K) $f(x) = (2x - 3)^2$ ....

(L) $f(x) = x^2 - x - 2$ ....

(M) $f(x) = x^2 - 8x + 16$. ....

(N) $f(x) = -4x^2 + 10x - 25$ ....

11) Graph the curve of the function $f(x) = x^2 - 9$ in the interval $[-3, 4]$, hence determine the sign of $f(x)$.

12) Graph the curve of the function $f(x) = -x^2 + 2x + 4$ in the interval $[-3, 5]$, hence determine the sign of $f(x)$.

13) **Discover the error:** If $f(x) = x + 1$, $g(x) = 1 - x^2$, then determine the interval at which the two functions are positive together .

yousof's answer

$x = -1$ makes $f(x) = 0$
$f(x)$ is positive in the interval $]-1, \infty[$,
$x = \pm 1$ makes $g(x) = 0$
$g(x)$ is positive in the interval $]-1, 1[$
thus the two functions are positive together in the interval
$]-1, \infty[\cup]-1, 1[=]-1, \infty[$

Amira's answer

$x = -1$ makes $f(x) = 0$
$f(x)$ is positive in the interval $]-1, \infty[$,
$x = \pm 1$ makes $g(x) = 0$
$g(x)$ is positive in the interval $]-1, 1[$
thus the two functions are positive together in the interval
$]-1, \infty[\cap]-1, 1[=]-1, 1[$

Which of the two answers is correct ? illustrate each of the two functions graphically and check your answer.

Quadratic Inequalities in one variable

1 - 5

You will Learn

- Solving the quadratic inequality in one variable.

Key - Terms

- Inequality

Materials

- Scientific calculator

Think and discuss

You have studied before the first degree inequality in one variable , and you have known that solving the inequality means, finding all values of the unknown which satisfy this inequality, and is written in the form of an interval. can you solve the second degree inequality in one variable?

Notice that:

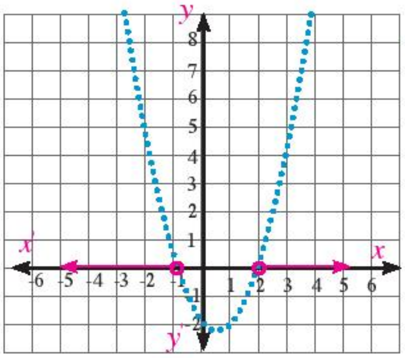
$x^2 - x - 2 > 0$ is a second degree inequality as shown in the figure opposite while $f(x) = x^2 - x - 2$

is the quadratic function related with this inequality.

From the figure opposite, we get:

➢ The solution set of the inequality $x^2 - x - 2 > 0$ in $\mathbb{R}$ is $]-\infty, -1[\cup]2, \infty[$

➢ The solution set of the inequality $x^2 - x - 2 < 0$ in $\mathbb{R}$ is $]-1, 2[$



Learn Solving the quadratic inequality in one variable

Example

(1) Solve the inequality: $x^2 - 5x - 6 > 0$

Solution

To solve the inequality, we do the following steps:

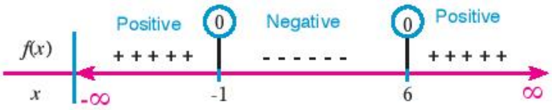
Step (1): We write the quadratic function related to the inequality as follows:

$f(x) = x^2 - 5x - 6$

Step (2): We study the sign of the function f where $f(x) = x^2 - 5x - 6$, and represent it on the number line by putting $f(x) = 0$

$x^2 - 5x - 6 = 0$ $\therefore (x - 6)(x + 1) = 0$

$x = 6$ or $x = -1$



Step (3): We determine the intervals which satisfy the inequality $x^2 - 5x - 6 > 0$

Then the solution set of the inequality is: $]-\infty, -1[\cup]6, \infty[$

Try to solve

1. Solve each of the following inequalities:

   (A) $x^2 + 2x - 8 > 0$ (B) $x^2 + x + 12 > 0$

Exercises (1 - 5)

Find the solution set of each of the following quadratic inequalities:

1. $x^2 \leqslant 9$
2. $x^2 - 1 \leqslant 0$
3. $2x - x^2 < 0$
4. $x^2 + 5 \leqslant 1$
5. $(x - 2)\,(x - 5) < 0$
6. $(x - 2)^2 \leqslant -5$
7. $x^2 > 6x - 9$
8. $3x^2 \leqslant 11x + 4$
9. $x^2 - 4x + 4 > 0$
10. $7 + x^2 - 4x < 0$

General Exercises

For more exercises, please visit the website of Ministry of Education.

Geometry

Unit 2

Similarity

Unit objectives

By the end of the unit, the student should be able to:

- Revise what he / she has previously studied in preparatory stage on similarity.
- Recognize similarity of two polygons.
- Recognize the theorem (if the length of the corresponding sides..)
- Recognize the theorem : (If the measure of an angle of a triangle ...)
- Recognize the theorem ; (the ratio between the areas of the surfaces of two similar triangle ...) .
- Recognize and deduce the fact: (Any two similar polygons can be divided into...)
- Recognize the theorem ; (the ratio between the areas of the surfaces of two similar polygons...)
- Recognize and deduce the well known problem ; (If the two lines containing the two chords in a circle are intersecting at a point then ...) and vice versa and corollaries on it .

Key - Terms

- Ratio
- Proportion
- Measure of an Angle
- Length
- Area
- Cross Product
- Extreme
- Mean
- Similar Polygons
- Similar Triangles
- Corresponding Sides
- Congruent Angles
- Regular Polygon
- Quadrilateral
- Pentagon
- Postulate/Axiom
- Perimeter
- Area of polygon
- Chord
- Secant
- Tangent
- Diameter
- Common External Tangent
- Common Internal Tangent
- Concentric Circles
- Similarity Ratio

Lessons of the Unit

Lesson (2 - 1): Similarity of Polygons.

Lesson (2 - 2): Similarity of Triangles.

Lesson (2 - 3): The Relation Between the Area of two Similar Polygons.

Lesson (2 - 4): Applications of Similarity in the circle.

Materials

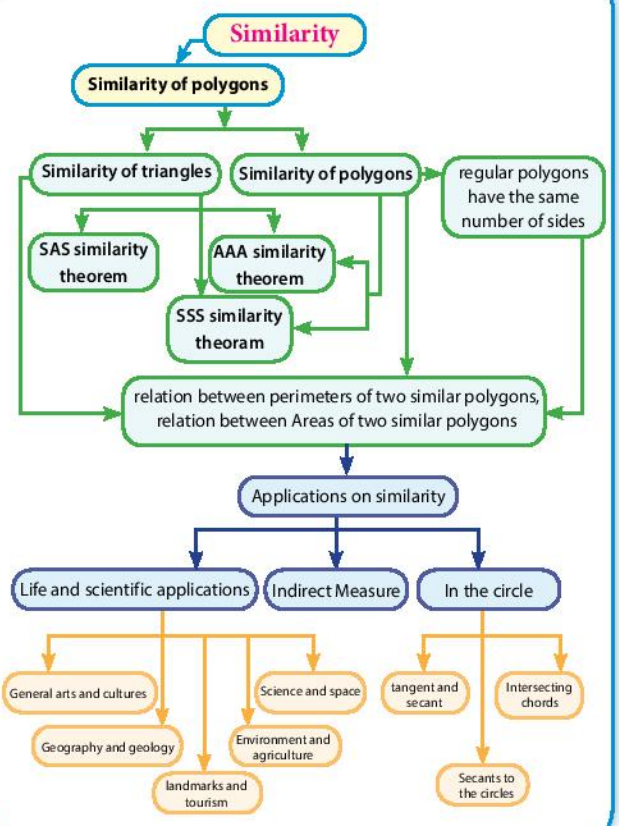
Computer – projector – graphic programs – squared paper – Mirror – Measuring tools – Calculator

Brief History

When construction on a piece of land, we need to do a sketch of the building. It is obvious that you can not do this sketch on a piece of paper to be congruent to the piece of land but we have to work on a small image similar to the original building. It could be done by taking a suitable scale for this minimization and measurements of angles on the drawing. So that they equal to their corresponding angles in the original building. If you think about the shape shown at the top of the page, you will notice that the nature is full of forms containing patterns repeat themselves at various scales, such as the leaves of a tree, flower head of cauliflower, and meanders see coast.

The notice of these repeated patterns led to the emergence of a new architecture for nearly 40 years, interested in self– study of shapes and summary that repeated irregular. It has been called fatafeet architecture or fractal engineering and it will be studied in the next stages of education.

Chart of the unit



2 - 1 Similarity of Polygons

You will learn

- Concept of similarity.
- Similarity of polygons
- The ratio between the perimeters of two similar polygon and the Drawing scale

Key-terms

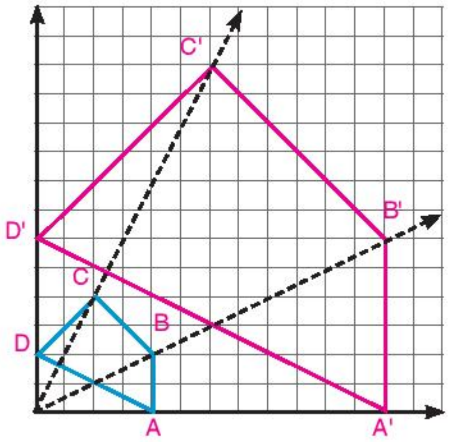
- Similar Polygons
- Similar Triangles
- Corresponding Sides
- Congruent Angles
- Regular Polygon
- Quadrilateral
- Pentagon
- Similarity Ratio

Learning tools

- Computer
- projector
- Graphic programs
- Squared paper
- Measuring tools
- Calculator

Think and discuss

the figure opposite shows the polygon ABCD, and its image A'B'C'D' by geometric transformation.



A Compare the measures of corresponding angles:
$\angle A$, $\angle A'$ - $\angle B$, $\angle B'$
$\angle C$, $\angle C'$ - $\angle D$, $\angle D'$
What do you deduce?

B Find the ratio of the lengths of the corresponding sides lengths $\frac{A'B'}{AB}, \frac{B'C'}{BC}, \frac{C'D'}{CD}, \frac{D'A'}{DA}$. What do you notice?

When the polygons having the same shape and different in lengths of sides, then they are called similar polygons.

Similar polygons

Definition **Two polygons, having the same number of sides, are said to be similar if their corresponding angles are congruent and the lengths of their corresponding sides are proportional.**

Notice that:

1- In the figure shown in "**think and discuss**" we get:

A **Corresponding angles are congruent:**

$\angle A' \equiv \angle A$, $\angle B' \equiv \angle B$
$\angle C' \equiv \angle C$, $\angle D' \equiv \angle D$

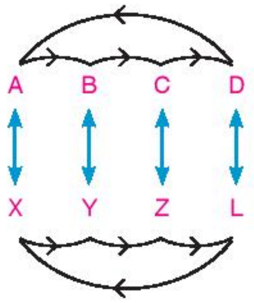
B **Lengths of corresponding sides are proportional:**

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{D'A'}{DA}$$

Thus, we can say that figure A'B'C'D' is similar to figure ABCD

2- We use the symbol (~) to denote similarity of two polygons and write them according to the order of their corresponding vertices to make it easy to write the proportion of corresponding sides.

If polygon ABCD ~ polygon XYZL then:



A $\angle A \equiv \angle X$, $\angle B \equiv \angle Y$, $\angle C \equiv \angle Z$, $\angle D \equiv \angle L$

B $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CD}{ZL} = \frac{DA}{LX} = K$ (similarity ratio), $K \neq 0$

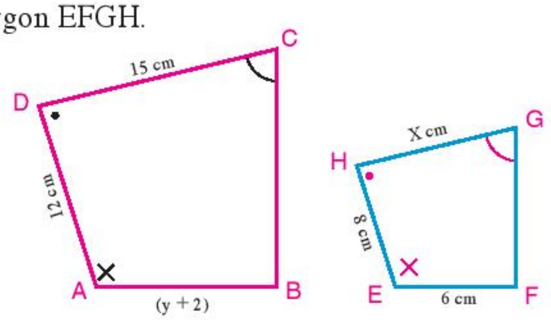
The scale factor of similarity of polygon ABCD to polygon XYZL equals K, and scale factor of similarity of polygon XYZL to polygon ABCD equals $\frac{1}{K}$

Example

1 **In the figure opposite**: polygon ABCD ~ polygon EFGH.

A Find the scale factor of similarity of polygon ABCD to polygon EFGH.

B Find the values of x and y.



Solution

$\because$ polygon ABCD ~ polygon EFGH

then: $\frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GH} = \frac{DA}{HE}$ = similarity ratio,

$$\frac{y+2}{6} = \frac{BC}{FG} = \frac{15}{x} = \frac{12}{8}$$

A Scale factor $= \frac{12}{8} = \frac{3}{2}$

B $\frac{15}{x} = \frac{3}{2} \longrightarrow x = 10\text{cm}$, $\frac{y+2}{6} = \frac{3}{2} \longrightarrow y = 7\text{cm}$

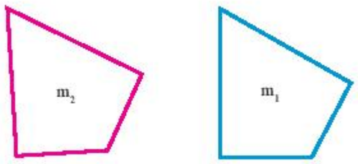
Think

Are all squares similar? Are all rhombuses similar?

Are all rectangles similar? Are all parallelogrames similar? Explain your answer.

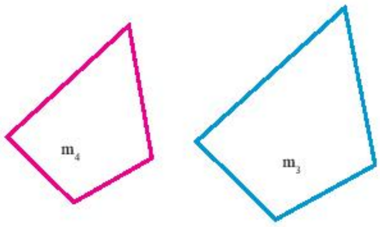
Notice that

1- For two polygons to be similar, the two conditions must be satisfied simultaneously, and the fulfilment of one of them only is not sufficient.



polygon $m_1 \equiv$ polygon m_2

2- If two polygons are congruent, then they are similar (polygon M_1 ~ polygon M_2) and the scale factor is then equal 1, and two simialr polygons need not be congruent (polygon $M_3 \not\equiv$ polygon M_4) as in the figure opposite.



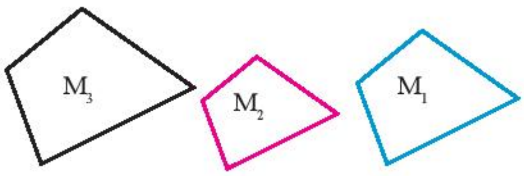
polygon m_3 ~ polygon m_4

3- If each one of two polygons is similar to a third polygon, then the two polygons are similar

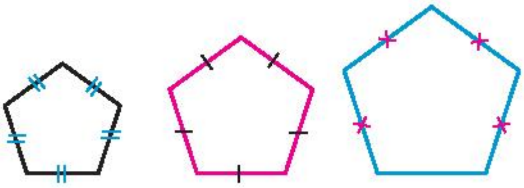
If polygon $M_1 \sim$ polygon M_3,

polygon $M_2 \sim$ polygon M_3

then: polygon $M_1 \sim$ polygon M_2



4- Any two regular polygons having the same number of sides are similar. Why?



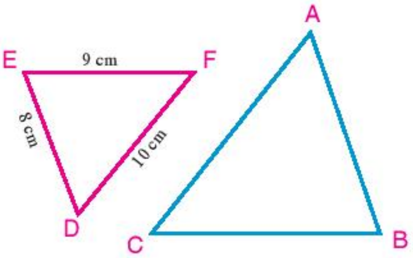
Example

2 **In the figure opposite**: $\triangle ABC \sim \triangle DEF$,

DE = 8cm , EF = 9cm , FD = 10cm

if the perimeter of $\triangle ABC$ = 81cm.

Find the side lengths of $\triangle ABC$.



Solution

$\because \triangle ABC \sim \triangle DEF$

$\therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{AB + BC + CA}{AE + EF + FD} = \frac{\text{perimeter of } \triangle ABC}{\text{perimeter of } \triangle DEF}$ (Proporites of proportion)

and: $\frac{AB}{8} = \frac{BC}{9} = \frac{CA}{10} = \frac{81}{27}$

$\therefore AB = 8 \times \frac{81}{27} = 24cm$, $BC = 9 \times 3 = 27$, $CA = 10 \times 3 = 30cm$

Notice that:

If polygon $M_1 \sim$ polygon M_2, then $\frac{\text{perimeter of } M_1}{\text{perimeter of } M_2}$ = Similarity ratio (scale factor)

Similarity ratio of two polygons

Let K be the similarity ratio of polygon M_1 to polygon M_2

If: $K > 1$ then polygon M_1 is an enlargement of polygon M_2

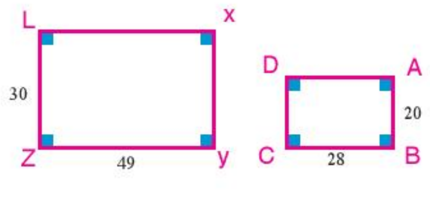
$0 < K < 1$ then polygon M_1 is a shrinking of polygon M_2

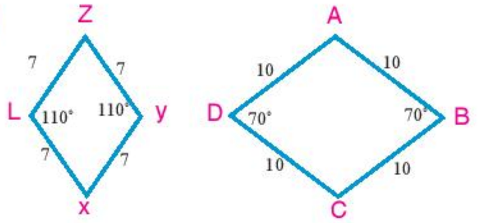
$K = 1$ then polygon M_1 is congruent to polygon M_2

In general: **you can use the similarity ratio in calculation of the dimensions of similar figures.**

Exercises (2 - 1)

1. Show which of the following pairs of polygons are similar, write the similar polygons in order of their corresponding vertices and determine the scale factor of similarity (side lengths are estimated in centimetres).

A


B


....

....

2. The dimensions of a rectangle are 10cm and 6cm. Find the perimeter and the area of another rectangle similar to it if:

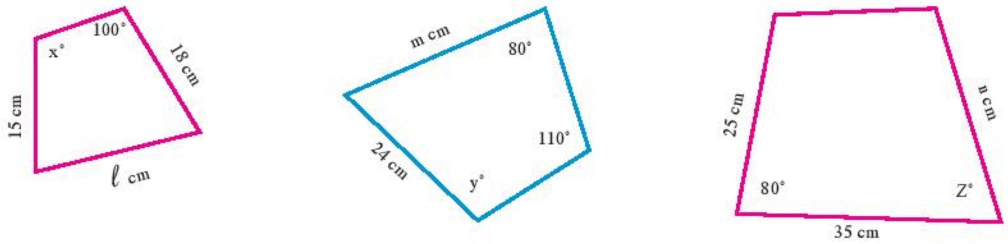
A Scale factor equals 3.

....

B Scale factor equals 0,4

....

3. The following three polygons are similar. Find the numerical value of the symbol used.



....

4. Two similar rectangles, the dimensions of the first are 8cm, 12cm and the perimeter of the second is 200cm. Find the length of the second rectangle and its area.

2 - 2 Similarity of Triangles

You will learn

- Cases of similarity of triangles.
- Properties of the perpendicular drawn from the vertex of the right angle to the hypotenuse of the right angled triangle

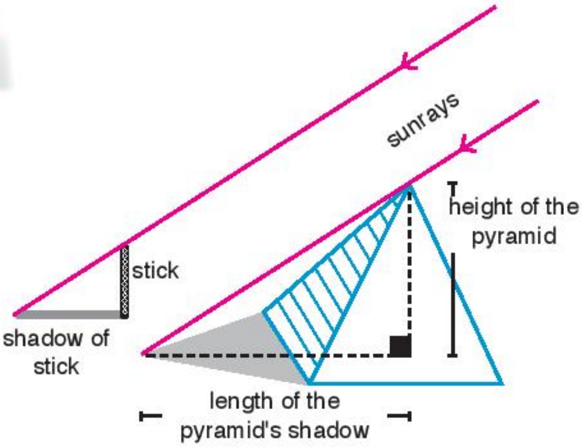
Key-terms

- Postulate/Axiom

Learning tools

- Computer
- projector
- Graphic programs
- Squared paper
- Mirror
- Measuring tools
- Calculator

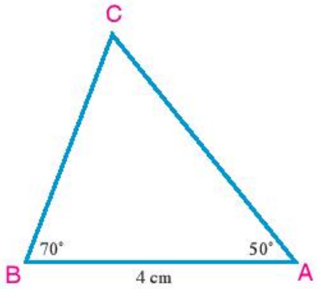
Think and discuss



One of the kings of pharaohs asked the mathematician THALES (600BC), to measure the height of the great pyramid, there were no equipments or machineries or a way to find the height of the pyramid directly. Thales fixed a vertical stick and start to measure the shadow of the stick and compare it with the length of the stick itself, until the length of the shadow of the stick equals the real length of the stick. He measured the length of the pyramid's shadow, it was equal to the height of the pyramid.

If you are asked to measure the height of a flagpole using a stick and a measuring tape. Do you wait until the length of the stick's shadow equals to the length of the stick or can you measure the flagpole's height at any time in a sunny day? Explain you answer.

Group work



1- Draw $\triangle ABC$ in which: $m(\angle A) = 50°$, $m(\angle B) = 70°$ and $AB = 4cm$

2- Draw $\triangle DEF$ in which: $m(\angle D) = 50°$, $m(\angle E) = 70°$, $DE = 5cm$

3- Find by measuring to the nearest millimetre, the length of: $\overline{AC}$, $\overline{BC}$, $\overline{DF}$ and $\overline{EF}$

4- Use the calculator to find the ratios $\frac{AC}{DF} = \frac{BC}{EF} = \frac{AB}{DE}$

Are the ratios equal? What do you deduce about these two triangles? Compare the results you obtained with the results obtained by the other groups and write your comment.

Postulate

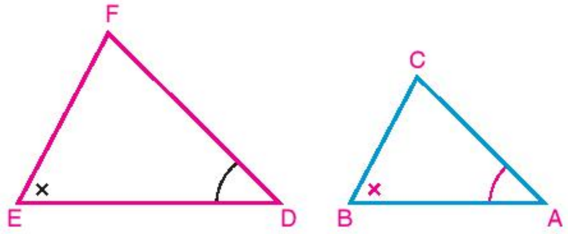
AA simiarity postulate

If two angles of one triangle are congruent to their two corresponding angles of another triangle, then the two triangles are similar

In the figure opposite:

If $\angle A \equiv \angle D$, $\angle B \equiv \angle E$

then $\triangle ABC \sim \triangle DEF$



Notice that

1- Any two equilateral triangles are similar

2- Two isosceles triangles are similar if the measure of one of the two base angles in one of them is equal to the measure of one of the two base angles in the other . or if the measures of their vertex angles are equal.

3- Two right triangles are similar if the measure of one of the two acute angles in one of them is equal to the measure of one of the two acute angles in the other.

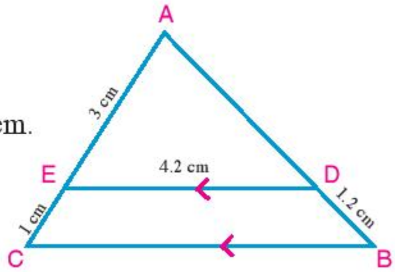
Example

(1) In the triangle ABC, $D \in \overline{AB}$ and $E \in \overline{AC}$ where $\overline{DE} \parallel \overline{BC}$,

BD = 1.2cm , AE = 3cm , AC= 4cm and DE = 4.2cm.

(A) Prove that $\triangle ADE \sim \triangle ABC$

(B) Find the length of: $\overline{AD}$ and $\overline{BC}$



Solution

(A) $\because \overline{DE} \parallel \overline{BC}$ and $\overleftrightarrow{AB}$ is a transversal to both of them.

$\therefore \angle ADE \equiv \angle ABC$

in the two triangles ADE, ABC

$\because \angle ADE \equiv \angle ABC$ (proved)

$\angle DAE \equiv \angle BAC$ (common angle)

$\therefore \triangle ADE \sim \triangle ABC$ (postulate)

(B) $\because \triangle ADE \sim \triangle ABC$

$\therefore \frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC}$ then,

$\frac{AD}{AD + 1.2} = \frac{3}{4} = \frac{4.2}{BC}$

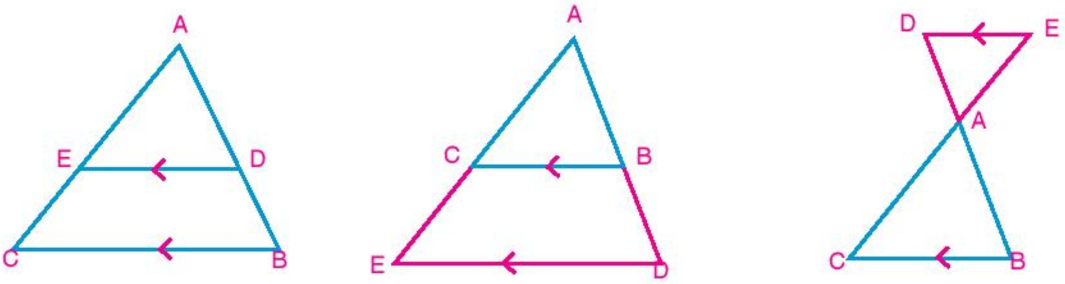
$4\,AD = 3(AD + 1.2)$, $3\,BC = 4 \times 4.2$

$= 3AD + 3.6$ $\qquad = \frac{4 \times 4.2}{3}$

$AD = 3.6cm$ $\qquad BC = 5.6cm$

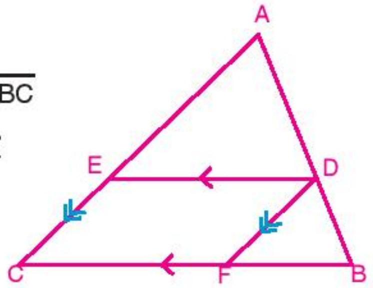
Important corollaries

corollary 1 **If a line is drawn parallel to one side of a triangle and intersects the other two sides or the lines containing them, then the resulting triangle is similar to the original triangle.**



If $\overleftrightarrow{DE} \parallel \overline{BC}$ and intersects $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ at D, and E respectively, as in the three previous figures:
Then: $\triangle ADE \sim \triangle ABC$.

Example



2 **In the figure opposite:** ABC is a triangle, $D \in \overline{AB}$, $\overrightarrow{DE} \parallel \overline{BC}$ and intersects $\overline{AC}$ at E, $\overrightarrow{DF} \parallel \overline{AC}$ and intersects $\overline{BC}$ at F.
Prove that: $\triangle ADE \sim \triangle DBF$

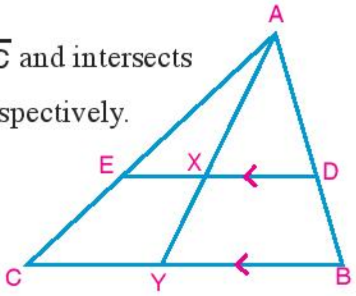
Solution

$\because \overline{DE} \parallel \overline{BC}$ $\quad \therefore \triangle ADE \sim \triangle ABC$ **(1)**

$\because \overline{DF} \parallel \overline{AC}$ $\quad \therefore \triangle DBF \sim \triangle ABC$ **(2)**

from **(1) and (2)**: $\triangle ADE \sim \triangle DBF$ **(Q.E.D)**

Try to solve

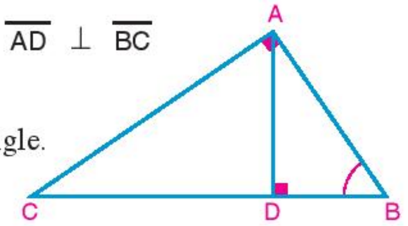


1 **In the figure opposite:** ABC is a triangle, $D \in \overline{AB}$, $\overrightarrow{DE} \parallel \overline{BC}$ and intersects $\overline{AC}$ at E, $\overrightarrow{AX}$ is drawn to intersect $\overline{DE}$ and $\overline{BC}$ at X and Y respectively.

A State three pairs of similar triangles.

B prove that: $\frac{DX}{BY} = \frac{XE}{YC} = \frac{DE}{BC}$.

corollary 2 **In any right triangle, the altitude to the hypotenuse separates the triangle into two triangles which are similar to each other and to the original triangle**



In the figure opposite: ABC is a right angled triangle at A, $\overline{AD} \perp \overline{BC}$

$\triangle$ DBA and $\triangle$ ABC

m ($\angle$ ADB) = m ($\angle$ CAB) = 90° and $\angle$ B is a common angle.

$\therefore \triangle$ DBA $\sim \triangle$ ABC (postulate) (1)

Similarly $\triangle$ DAC $\sim \triangle$ ABC (2)

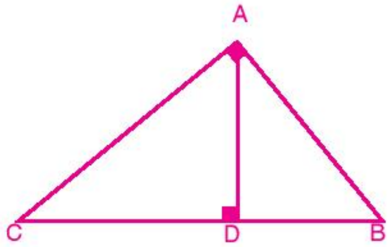
$\because$ If each one of two triangles is similar to a third triangle, then the two triangles are similar.

$\therefore \triangle$ DBA $\sim \triangle$ DAC $\sim \triangle$ ABC

Example

3 ABC is a right angled triangle at A, $\overline{AD} \perp \overline{BC}$. Prove that DA is a mean proportional to DB and DC.

Solution



Given: $\triangle$ ABC: m ($\angle$A) = 90°, $\overline{AD} \perp \overline{BC}$.

R.T.P: prove that $(DA)^2 = DB \times DC$.

Proof: In $\triangle$ ABC

$\because$ m ($\angle$A) = 90°, $\overline{AD} \perp \overline{BC}$

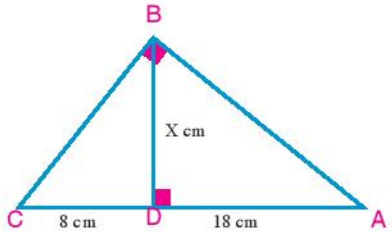
$\therefore \triangle$ DBA $\sim \triangle$ DAC (Corollary)

and $\frac{DA}{DC} = \frac{DB}{DA}$ i.e. $(DA)^2 = DB \times DC$

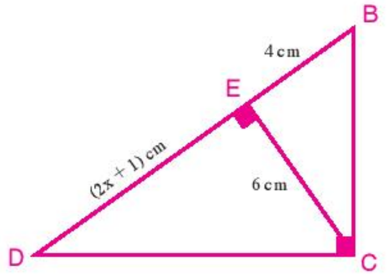
Try to solve

2 In each of the following figures, Find the numerital value of x:

A

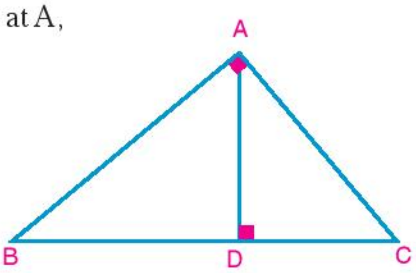


B



Example

4 **In the figure opposite:** ABC is a right angled triangle at A, $\overline{AD} \perp \overline{BC}$ prove that:



A $(AB)^2 = BC \times BD$

B $(AC)^2 = CB \times CD$

Solution

In △ ABC:

$\because m(\angle A) = 90°$, $\overline{AD} \perp \overline{BC}$

$\therefore \triangle ABD \sim \triangle CBA$ (corollary)

$\therefore \frac{AB}{CB} = \frac{BD}{BA}$, $(AB)^2 = BD \times BC$

, $\triangle ACD \sim \triangle BCA$ (corollary)

$\therefore \frac{AC}{BC} = \frac{CD}{CA}$, $(AC)^2 = CB \times CD$

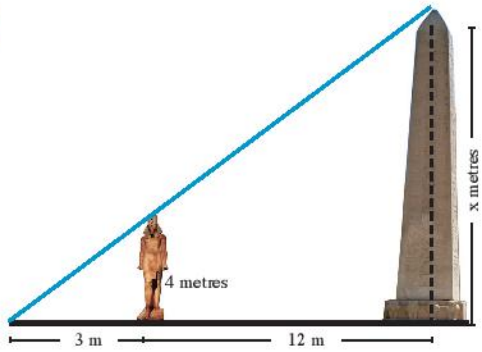
add to your knowledge

The results you obtained in exmples 3, 4 are considederd a proof of Euclid theorem which you studied in the preperatory stage

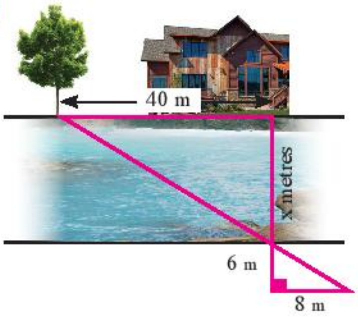
Try to solve

3 Find the distance x in each of the following cases:

A



B



theorem 1

sss Similarity theorem *(Proof is not required)*

If the side lengths of two triangles are in proportion, then the two triangles are similar.

Given: In △△ ABC and DEF: $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$

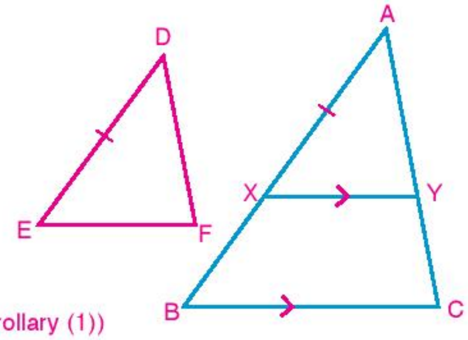
R.t.p. : $\triangle ABC \sim \triangle DEF$

Proof : take $X \in \overline{AB}$ where AX = DE,

Draw $\overline{XY} \parallel \overline{BC}$ and intersects $\overline{AC}$ at Y.

$\because \overline{XY} \parallel \overline{BC}$

$\therefore \triangle ABC \sim \triangle AXY$ (Corollary (1))



$\therefore \frac{AB}{AX} = \frac{BC}{XY} = \frac{CA}{YA}$

$\because AX = DE$ (Construction)

$\therefore \frac{AB}{DE} = \frac{BC}{XY} = \frac{CA}{YA}$ (1)

$\because \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$ (given) (2)

from (1), (2) we deduce that: XY = EF, YA = FD

and $\triangle AXY \equiv \triangle DEF$ (SSS congruency theorem)

$\therefore \triangle DEF \sim \triangle AXY$

$\because \triangle ABC \sim \triangle AXY$ (Proved)

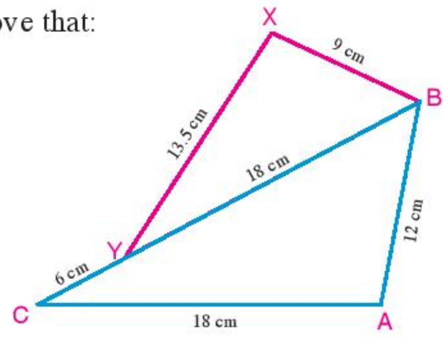
$\therefore \triangle ABC \sim \triangle DEF$ (Q.E.D.)

Example

5 **In the figure opposite:** B, Y and C are collinear. Prove that:

A $\triangle ABC \sim \triangle XBY$

B $\overrightarrow{BC}$ bisects $\angle ABX$



Solution

A In the two triangles ABC and XBY, we have:

$\frac{AB}{XB} = \frac{12}{9} = \frac{4}{3}$, $\frac{BC}{BY} = \frac{18+6}{18} = \frac{4}{3}$

$\frac{AC}{XY} = \frac{18}{13.5} = \frac{4}{3}$

, $\frac{AC}{XY} = \frac{BC}{BY} = \frac{AB}{XB}$ i.e. Corresponding sides are proportional

$\therefore \triangle ABC \sim \triangle XBY$

B $\because \triangle ABC \sim \triangle XBY$ $\therefore m(\angle ABC) = m(\angle XBY)$

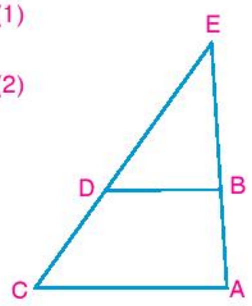
i.e.: $\overrightarrow{BC}$ bisects $\angle ABX$

6 **In the figure opposite:** $\overrightarrow{AB} \cap \overrightarrow{CD} = \{E\}$ where $\frac{AE}{CE} = \frac{BE}{DE}, \frac{AC}{CE} = \frac{BD}{DE}$ prove that $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$

Solution

$\because \frac{AE}{CE} = \frac{BE}{DE}$ $\therefore \frac{AE}{BE} = \frac{CE}{DE}$ (Properties of proportion) (1)

$\because \frac{AC}{CE} = \frac{BD}{DE}$ $\therefore \frac{AC}{BD} = \frac{CE}{DE}$ (Properties of proportion) (2)



From (1), (2) we get: $\frac{AE}{BE} = \frac{CE}{DE} = \frac{CA}{DB}$

i.e. $\triangle AEC \sim \triangle BED$

$\therefore m(\angle ACE) = m(\angle BDE)$

They are corresponding w.r. to the transversal $\overleftrightarrow{CE}$

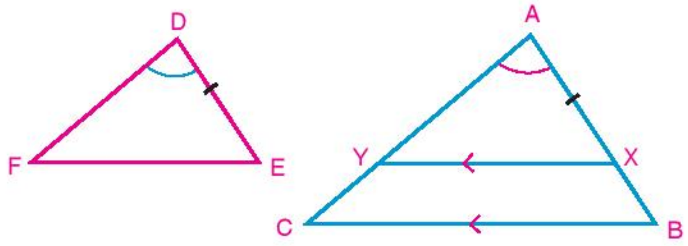
$\therefore \overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$

theorem 2

SAS Similarity theorem *(Proof is not required)*

If an angle of one triangle is congruent to an angle of another triangle and lengths of the sides including those angles are in proportion, then the triangles are similar

Given: $\angle A \equiv \angle D$, $\frac{AB}{DE} = \frac{AC}{DF}$



R.t.p.: $\triangle ABC \sim \triangle DEF$

Proof: take $X \in \overline{AB}$ where $AX = DE$

draw $\overrightarrow{XY} \parallel \overline{BC}$

and intersect $\overline{AC}$ at Y

$\because \overline{XY} \parallel \overline{BC}$ $\quad \therefore \triangle ABC \sim \triangle AXY$ (Corollary) (1)

, $\frac{AB}{AX} = \frac{AC}{AY}$

$\because \frac{AB}{DE} = \frac{AC}{EF}$ (given) , $AX = DE$ (construction)

$\therefore \frac{AB}{AX} = \frac{AC}{EF}$, $AY = DF$

$\therefore \triangle AXY \equiv \triangle DEF$ (SAS congruency theorem)

$\triangle AXY \sim \triangle DEF$ (2)

from (1), (2) we get: $\triangle ABC \sim \triangle DEF$ Q.E.D.

Example

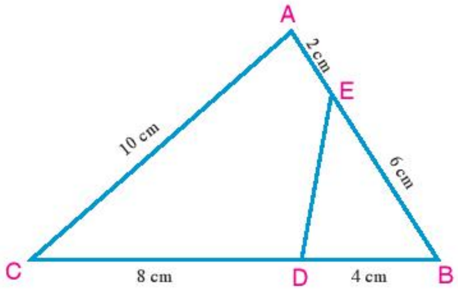
7) ABC is a triangle AB = 8cm, AC = 10cm, BC = 12cm, $E \in \overline{AB}$ where AE = 2cm, $D \in \overline{BC}$ where BD = 4cm.

A) Prove that $\triangle BDE \sim \triangle BAC$ and deduce the length of $\overline{DE}$.

B) Prove that figure ACDE is a cyclic quadrilateral.

Solution

$\because AB = 8cm, AE = 2cm$ $\quad \therefore BE = 6cm$



A) In $\triangle\triangle$ BDE, BAC,:

$\angle DBE \equiv \angle ABC$ (1)

, $\frac{BD}{BA} = \frac{4}{8} = \frac{1}{2}$, $\frac{BE}{BC} = \frac{6}{12} = \frac{1}{2}$

$\therefore \frac{BD}{BA} = \frac{BE}{BC}$ (2)

from (1), (2) $\therefore \triangle BDE \sim \triangle BAC$ (theorem)

from similarity $\frac{DE}{AC} = \frac{1}{2}$

$\therefore DE = \frac{1}{2} AC$, $DE = \frac{1}{2} \times 10 = 5cm$

B From similarity also $\angle$ BDE $\equiv$ $\angle$ BAC $\quad\therefore$ m($\angle$ BDE) = m($\angle$ BAC)

$\because$ $\angle$ BDE is an exterior angle of the quadrilateral ACDE

$\therefore$ the figure ACDE is a cyclic quadrilateral.

Example

8 ABC is a triangle D $\in$ $\overline{BC}$ where $(AC)^2 = CD \times CB$. Prove that: $\triangle$ ACD $\sim$ $\triangle$ BCA

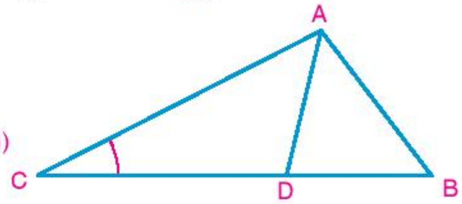
Solution

In the two triangles ABC, DAC, $\angle$C is a common angle (1)

$\because (AC)^2 = CE \times CB$

$\therefore \frac{AC}{CB} = \frac{CD}{AC}$ (2)

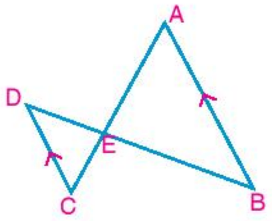
from (1), (2) we get $\triangle$ ACD $\sim$ $\triangle$ BCA (theorem)



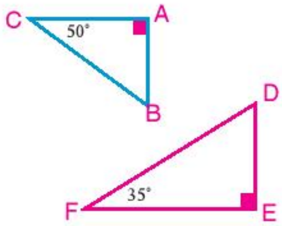
Exercises (2 - 2)

1 State which of the following cases, the two triangles are similar. In case of similarity, state why they are similar?

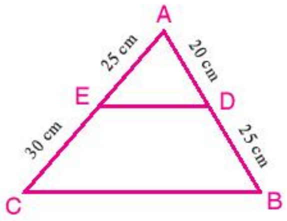
A



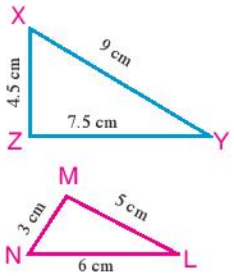
B



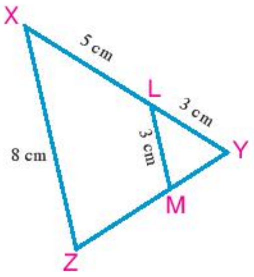
C



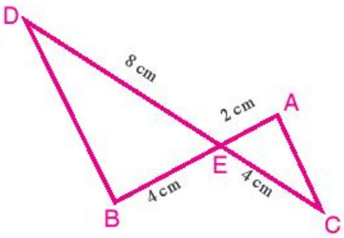
D



E

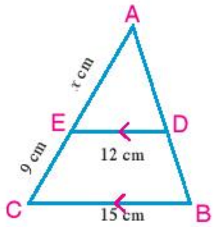


F

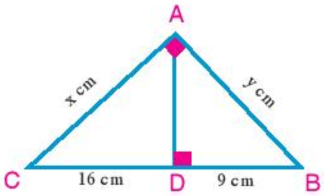


2 Find the value of the symbol used:

A



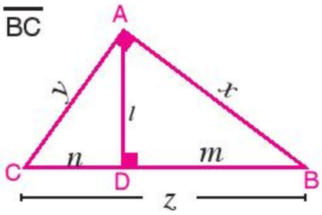
B



3 **In the figure opposite:** ABC is a right angled triangle, $\overline{AE} \perp \overline{BC}$

First: complete: $\triangle ABC \sim \triangle$ .... $\sim \triangle$ ....

Second: If x, y, z, l, m and n are the lengths of the straight segments in centimetres, then complete the following proportions:



A $\frac{x}{z} = \frac{m}{\dots}$ **B** $\frac{x}{z} = \frac{l}{\dots}$ **C** $\frac{m}{l} = \frac{x}{\dots}$ **D** $\frac{l}{\dots} = \frac{\dots}{l}$

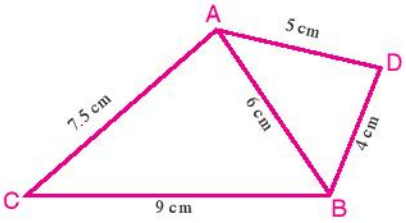
E $\frac{x}{\dots} = \frac{\dots}{x}$ **F** $\frac{\dots}{y} = \frac{y}{\dots}$ **G** $\frac{l}{x} = \frac{\dots}{z}$ **H** $\frac{l}{x} = \frac{\dots}{y}$

4 $\overline{AB}$ and $\overline{DC}$ are two chords in a circle, $\overrightarrow{AB} \cap \overrightarrow{DC} = \{E\}$, where E lies outside the circle, AB = 4cm, DC = 7cm and BE = 6cm. prove that $\triangle ADE \sim \triangle CBE$, then find the length of $\overline{CE}$

5 In $\triangle$ ABC, AC < AB, M $\in \overline{AC}$ where m($\angle$ABM) = m ($\angle$C).
Prove that $(AB)^2 = AM \times AC$.

6 ABC is a right angled triangle at A, $\overrightarrow{AD} \perp \overline{BC}$ to intersect it at D . if $\frac{BD}{DC} = \frac{1}{2}$, $AD = 6\sqrt{2}$ cm. Find the length of $\overline{BD}$, $\overline{AB}$ and $\overline{AC}$.

7 **In the figure opposite:** A B C is a triangle in which AB = 6cm , BC = 9cm and AC = 7,5cm.
D is a point outside the triangle ABC
where D B = 4cm and DE = 5cm. Prove that:



A $\triangle ABC \sim \triangle DBA$

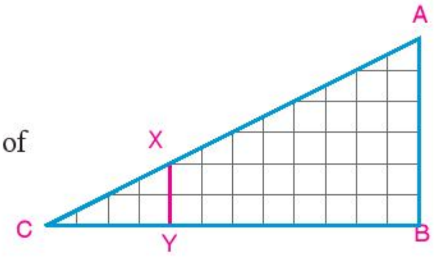
B $\overrightarrow{BA}$ bisects $\angle$ DBC

The Relation Between the Areas of two Similar Polygons

2 - 3

Think and discuss

On the squared paper, draw each of the two triangles ABC and XYC.



1- Show why:
$\triangle XYC \sim \triangle ABC$. Find the similarity ratio.

2- Calculate the ratio of area of the triangle XYC to the area of the original triangle ABC

3- Determine another point as $D \in \overline{AC}$, then draw $\overrightarrow{DD'} \parallel \overline{AB}$ and intersects $\overline{BC}$ at D' to get the triangle DD'C. Is $\triangle DD'C \sim \triangle XYC$?

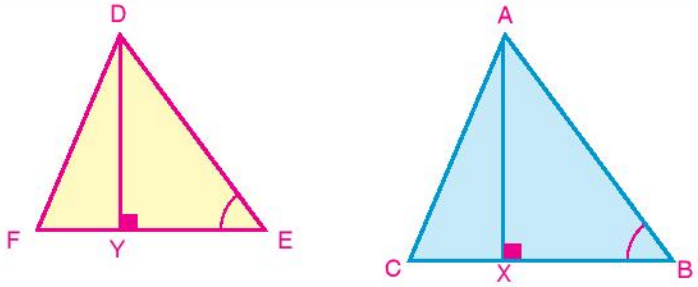
4- Complete the following table:

The triangles	similarity ratio	Area of the first triangle	Area of the second triangle	Ratio between their areas
$\triangle XYC \sim \triangle ABC$	$\frac{1}{3}$	4	36	$\frac{4}{36} = \frac{1}{9}$
$\triangle DD'C \sim \triangle ABC$				
$\triangle XYC \sim \triangle DD'C$				

5- What does it mean by the ratios you obtained comparing with the similarity ratio (scale factor)?

First: the ratio of Areas of two similar triangles:

Theorem 3 **Ratio of the areas of the surfaces of two similar triangles equals the square of the ratio of the lengths of any two corresponding sides of the two triangles** *(Proof is not required)*



You will learn

- The relation between the perimeters of two similar polygons and similarity ratio (scale factor)
- The relation between areas of two similar polygons and similarity ratio.

Key-term

- Perimeter
- Area
- Area of a Polygon
- Corresponding Sides

learning tools

- Computer
- Projector
- Graphic program
- squared paper
- Calculator

Given: $\triangle ABC \sim \triangle DEF$

R.t.p.: $\frac{a(\triangle ABC)}{a(\triangle DEF)} = \left(\frac{AB}{DE}\right)^2 = \left(\frac{BC}{EF}\right)^2 = \left(\frac{CA}{FD}\right)^2$

> **Notice**
> the symbol (a) expresses the surface area of a polygon

Proof: Draw $\overrightarrow{AX} \perp \overline{BC}$ where $\overrightarrow{AX} \cap \overline{BC} = \{X\}$,

$\overrightarrow{DY} \perp \overline{EF}$ where $\overrightarrow{DY} \cap \overline{EF} = \{Y\}$

$\because \triangle ABC \sim \triangle DEF$

$\therefore m(\angle B) = m(\angle E)$, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$ (1)

In the two triangles ABX, DEY:

$m(\angle X) = m(\angle Y) = 90°$, $m(\angle B) = m(\angle E)$

$\therefore \triangle ABX \sim \triangle DEY$ (A A similarity postulate)

$\therefore \frac{AB}{DE} = \frac{AX}{DY}$ (2)

$$\frac{a(\triangle ABC)}{a(\triangle DEF)} = \frac{\frac{1}{2} BC \times AX}{\frac{1}{2} EF \times AY} = \frac{BC}{EF} \times \frac{AX}{DY}$$

By substituting from (1), (2), we get:

$$\frac{a(\triangle ABC)}{a(\triangle DEF)} = \frac{AB}{DE} \times \frac{AB}{DE} = \left(\frac{AB}{DE}\right)^2 = \left(\frac{BC}{EF}\right)^2 = \left(\frac{CA}{FE}\right)^2$$

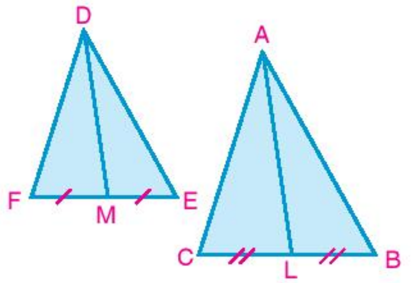
Q.E.D.

Notice that: $\frac{a(\triangle ABC)}{a(\triangle DEF)} = \left(\frac{AB}{DE}\right)^2$, $\frac{AB}{DE} = \frac{AX}{DY}$

then: $\frac{a(\triangle ABC)}{a(\triangle DEF)} = \left(\frac{AX}{DY}\right)^2$

i.e. **the ratio of the areas of the surfaces of two similar triangles equals the square of the ratio the lengths of any two corresponding alttitudes of the two triangles.**

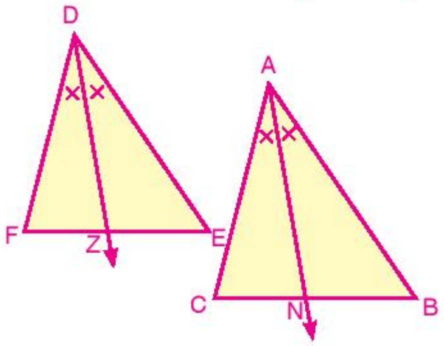
Critical thinking:



1- If $\triangle ABC \sim \triangle DEF$, L is the midpoint of $\overline{BC}$, M is the midpoint of $\overline{EF}$.

Is $\frac{a(\triangle ABC)}{a(\triangle DEF)} = \left(\frac{AL}{DM}\right)^2$?

Explain your answer, and write your deduction.



2- If $\triangle ABC \sim \triangle DEF$,

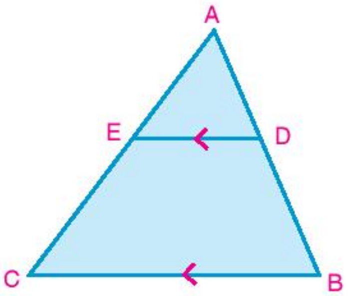
$\overrightarrow{AN}$ bisects $\angle A$ and intersects $\overline{BC}$ at N,

$\overrightarrow{DZ}$ bisects $\angle D$ and intersects $\overline{EF}$ at Z.

Is $\frac{a(\triangle ABC)}{a(\triangle DEF)} = \left(\frac{AN}{DZ}\right)^2$?

Explain your answer and write your deduction.

Example



1 **In the figure opposite:** ABC is a triangle D ∈ $\overline{AB}$ where $\frac{AD}{DB} = \frac{3}{4}$, $\overrightarrow{DE}$ // $\overline{BC}$ and intersects $\overline{AC}$ at E.

if the area of △ ABC = 784cm². find:

A area of △ ADE. **B** area of trapezium DBCE.

Solution

In △ADC: ∵ $\overline{DE}$ // $\overline{BC}$

∴ △ ADE ~ △ ABC (Corollary)

∴ $\frac{a(\triangle ADE)}{a(\triangle ABC)} = \left(\frac{AD}{AB}\right)^2$ (theorem)

and $\frac{a(\triangle ADE)}{784} = \left(\frac{3}{7}\right)^2$ ∴ a(△ ADE) = $784 \times \frac{9}{49} = 144cm^2$

∵ area of trapezium DBCE = area of △ ABC – area of △ ADE

∴ area of trapezium DBCE = 784 – 144 = 640cm²

Example

2 The ratio of the areas of two similar triangles equals 4 : 9. If the perimeter of the greater triangle equals 90cm, Find the perimeter of the smaller triangle.

Solution

Let △ ABC ~ △ DEF

∴ $\frac{a(\triangle ABC)}{a(\triangle DEF)} = \left(\frac{AB}{DE}\right)^2 = \frac{4}{9}$ $\quad \frac{AB}{DE} = \frac{2}{3}$

∵ $\frac{\text{Perimeter of } \triangle ABC}{\text{Perimeter of } \triangle DEF} = \frac{AB}{DE} = \frac{2}{3}$

∴ $\frac{\text{Perimeter of } (\triangle ABC)}{90} = \frac{2}{3}$

∴ Perimeter of △ ABC = 60cm

Try to solve

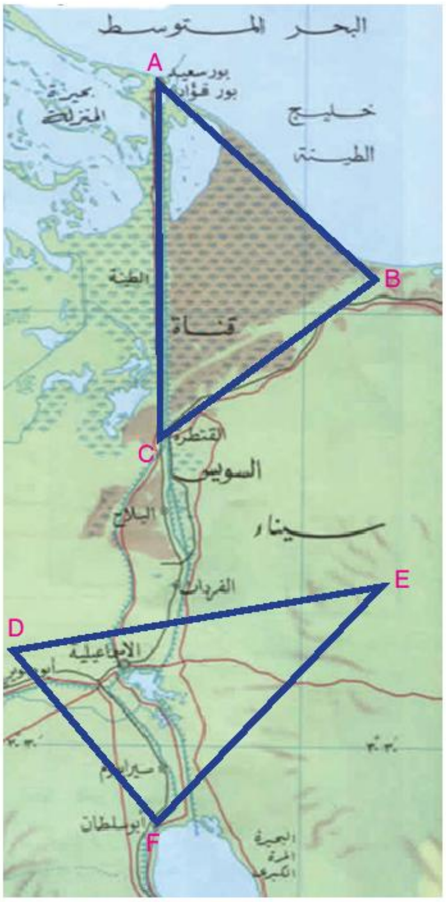
1 ABC and DEF are two similar triangles, $\frac{a(\triangle ABC)}{a(\triangle DEF)} = \frac{3}{4}$

A If the perimeter of smaller triangle equals $45\sqrt{3}$ cm. Find the perimeter of the greater triangle.

B If EF = 28cm, find the length of $\overline{BC}$.

Example

3 If every 1 cm on the map represents 10 kilometres. find the real area which the triangle ABC represents to the nearest square kilometre,
If $a(\triangle ABC) = 6.4cm^2$.



Solution

Drawing scale = similarity ratio = $\frac{1}{10 \times 10^5}$

$\frac{\text{area of } \triangle ABC}{\text{real area}}$ = square of similarity ratio

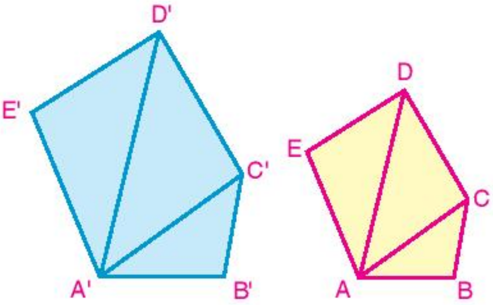
$\frac{6.4}{\text{real area}} = \left(\frac{1}{10 \times 10^5}\right)^2$

the real area $= 6.4 \times 10 \times 10 \times 10^5 \times 10^5 \text{ cm}^2$

$\simeq 640 \text{ km}^2$

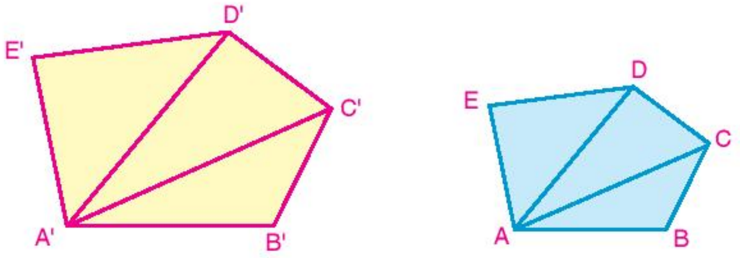
Fact: Two similar polygon can be divided into the same number of triangles, each is similar to the corresponding one.

Note: the fact mentioned above is valid regardless the number of the sides of the two similar polygons (have always the same number of sides). Any polygon of n sides can be divided into (n–2) triangles.



theorem 4 **Ratio of the areas of the surfaces of two similar polygons equals the square of the ratio of the lengths of any two corresponding sides of the polygons**

(Proof is not required)



Given: polygon ABCDE ~ polygon A'B'C'D'E'

R.t.p.: $\frac{a\text{ (polygon ABCDE)}}{a\text{ (polygon A'B'C'D'E')}} = \left(\frac{AB}{A'B'}\right)^2$

proof: from A, A' draw $\overline{AC}$, $\overline{AD}$, $\overline{A'C'}$, $\overline{A'D'}$

$\because$ polygon ABCDE ~ polygon A'B'C'D'E'

$\therefore$ they are divided into the same number of triangles, each is similar to the corresponding one **(fact)**. then:

$\frac{a(\triangle ABC)}{a(\triangle A'B'C')} = \left(\frac{BC}{B'C'}\right)^2$, $\frac{a(\triangle ADE)}{a(\triangle A'D'E')} = \left(\frac{CD}{C'D'}\right)^2$, $\frac{a(\triangle ACD)}{a(\triangle A'CD')} = \left(\frac{DE}{D'E'}\right)^2$

$\because \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{AB}{A'B'}$ (from similar polygons)

$\therefore \frac{a(\triangle ABC)}{a(\triangle A'B'C')} = \frac{a(\triangle ACD)}{a(\triangle A'C'D')} = \frac{a(\triangle ADE)}{a(\triangle A'D'E')} = \left(\frac{AB}{A'B'}\right)^2$

From properties of proportion

$$\frac{a(\triangle ABC) + a(\triangle ACD) + a(\triangle ADE)}{a(\triangle A'B'C') + a(\triangle A'C'D') + a(\triangle A'D'E')} = \left(\frac{AB}{A'B'}\right)^2$$

then: $\frac{a\text{ (polygon ABCDE)}}{a\text{ (polygon A'B'C'D'E')}} = \left(\frac{AB}{A'B'}\right)^2$ Q.E.D

Notice

$\left(\frac{AB}{A'B'}\right)^2 = \frac{(AB)^2}{(A'B')^2}$

Try to solve

2 **A** If polygon ABCD ~ polygon A'B'C'D', $\frac{AB}{A'B'} = \frac{1}{3}$, then write the value of each of the following:

$\frac{a\text{ (polygon ABCD)}}{a\text{ (polygon A'B'C'D')}}$, $\frac{\text{perimeter of polygon ABCD}}{\text{perimeter of polygon A'B'C'D'}}$

B If the two polygons ABCDE and A'B'C'D'E' are similar and the ratio of their areas equals 4 : 25

Then write the value of each of: $\frac{AB}{A'B'}$, $\frac{\text{perimeter of polygon ABCDE}}{\text{perimeter of polygon A'B'C'D'E'}}$

C If the ratio of the perimeters of two similar polygons equals 1 : 4, and area of the first polygon equals $25cm^2$. Find the area of the second polygon.

D If the length of two corresponding sides in two similar polygons are 12cm and 16cm, and the area of the smaller polygon equals $135cm^2$. Then find the area of the greater polygon.

Example

4 ABCD and XYZL are two similar polygons and : $m(\angle A) = 40°$, $XY = \frac{3}{4}AB$, CD = 16cm.

Calculate: **first:** $m(\angle X)$

Second: length of $\overline{ZL}$

third: a(polygon ABCD) : a (polygon X Y Z L)

Solution

$\because$ polygon ABCD ~ polygon XYZL

$\therefore m(\angle A) = m(\angle X)$ then $m(\angle X) = 40°$ (first required)

$\because XY = \frac{3}{4}AB$ $\therefore \frac{AB}{XY} = \frac{4}{3}$ (from properties of proportion)

From similar polygons, we get also $\frac{AB}{XY} = \frac{CD}{ZL}$

$\therefore \frac{4}{3} = \frac{16}{ZL}$ then $ZL = \frac{3 \times 16}{4} = 12cm$ (second required)

a(polygon ABCD): a (polygon XYZL) $= (AB)^2 : (XY)^2$

$= 16k^2 : 9k^2$

$= 16 : 9$ (third required)

Notice that

AB = 4K

XY = 3K

$K \neq 0$

Example

5 The ratio of the perimeters of two similar polygons equals 3 : 4, if the sum of their areas equals $225cm^2$, then find the area of each of them.

Solution

$\because$ The ratio of the perimeters of the two similar polygons = 3 : 4

$\therefore$ The ratio of the lengths of any two corresponding sides of them = 3 : 4

Let the area of the first polygon be 9x ,

and the area of the second polygon be 16x

$\therefore$ $9x + 16x = 225$, then $x = \frac{225}{9 + 16} = 9$

$\therefore$ area of the first polygon = $9 \times 9 = 81cm^2$

$\therefore$ area of the second polygon = $16 \times 9 = 144cm^2$

Try to solve

3 **Agriculture:** Two farms are in the form of similar polygons, the ratio of the lengths of two corresponding sides of them equals 5:3, if the difference between their areas equals 32 feddans, then find the area of each farm.

Exercises (2 - 3)

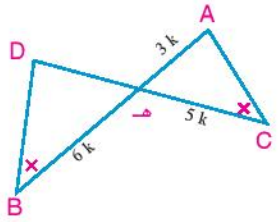
1 Complete:

A If $\triangle ABC \sim \triangle XYZ$, and $AB = 3\,XY$, then $\frac{\text{area}(\triangle XYZ)}{\text{area}(\triangle ABC)} = $ ....

B If $\triangle ABC \sim \triangle DEF$, area of $(\triangle ABC) = 9$ area of $(\triangle DEF)$ and $DE = 4$cm, then $AB = $ .... cm

2 Study each of the following figures, where K is constant of proportion , then complete:

A

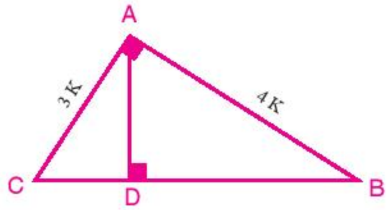


$\overline{AB} \cap \overline{CD} = \{E\}$

area of $(\triangle ACE) = 900\ cm^2$

then: area of $(\triangle DEB) = $ .... cm^2

B



$m(\angle BAC) = 90^\circ$, $\overline{AD} \perp \overline{BC}$

area of $(\triangle ADC) = 180\ cm^2$ then:

area of $(\triangle ABC) = $ .... cm^2

3 ABC is a triangle, $D \ni \overline{AB}$ where $AD = 2\,BD$ and $E \in \overline{AC}$ where $\overline{DE} \mathbin{/\!/} \overline{BC}$

If the area of $\triangle ADE = 60cm^2$, find the area of the trapezium DBCE.

....

....

4 ABC is a right angled triangle at B. The equilateral triangles ABX, BCY, and ACZ are drawn . prove that : area of $(\triangle ABX)$ + area of $(\triangle BCY)$ = area of $(\triangle ACZ)$.

....

....

5 ABC is an inscribed triangle in a circle where $\frac{AB}{BC} = \frac{4}{3}$. from the point B, a tangent is drawn to the circle and intersects $\overrightarrow{AC}$ at E.

prove that: $\frac{\text{area of }(\triangle ABC)}{\text{area of }(\triangle ABE)} = \frac{7}{16}$

....

....

6 ABCD is a parallelogram, $X \in \overrightarrow{AB}$, $X \notin \overline{AB}$, where $BX = 2\,AB$, $Y \in \overrightarrow{CB}$, $Y \notin \overline{CB}$,

where $BY = 2\,BC$. the parallelogram BXZY is drawn, prove that: $\frac{\text{area of (ABCD)}}{\text{area of (XBYZ)}} = \frac{1}{4}$

....

....

2 - 4

Applications of Similarity in the circle

You will learn

- The relation between two intersecting chords in a circle.
- The relation between two secants to the circle from a point outside it.
- The relation between the length of a tangent and the lengths of the two parts of a secant to the circle drawn from a point outside it.
- Modeling, solving problems, and life applications using similiarity of polygons in a circle.

Key-term

- Chord
- Secant
- Tangent
- Diameter
- Common External Tangent
- Common Internal Tangent
- Concentric Circles

Think and discuss

In each of the following figures, two similar triangles. Write the two triangles in order of their corresponding congruent angles, and deduce the proportion of their corresponding sides.

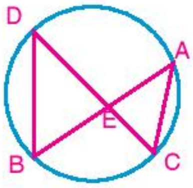


figure (1)

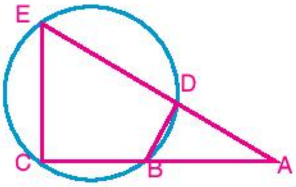


figure (2)

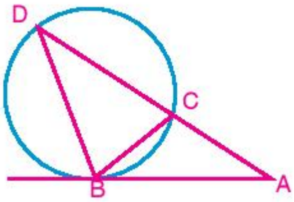


figure (3)

- **In figure (1):** Is there a relation between EA × EB and EC × ED?
- **In figure (2):** Is there a relation between AE × AD and AC × AB?
- **In figure (3):** Is there a relation between AD × AC and $(AB)^2$?

Well known problem

If the two lines containing the two chords $\overline{AB}$, $\overline{CD}$ of a circle are intersecting at the point E, then EA × EB = EC × ED

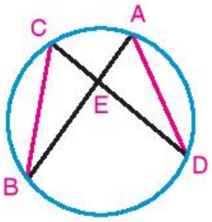


figure (1)

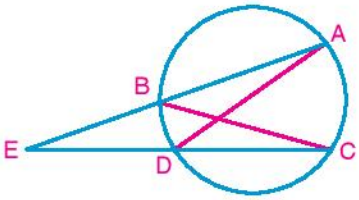


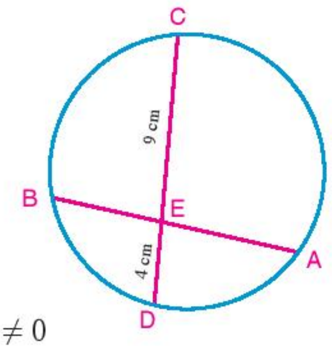
figure (2)

To deduce that:

- Draw $\overline{AD}$ and $\overline{BC}$
- In each of the two figures, prove that the two triangles are similar:

then $\frac{EA}{EC} = \frac{ED}{EB}$ $\therefore$ EA × EB = EC × ED

Example



1 **In the figure opposite:** $\overline{AB} \cap \overline{CD} = \{E\}$,

if $\frac{EA}{EB} = \frac{4}{3}$, EC = 9cm and ED = 4cm

Find the length $\overline{EB}$

Solution

$\because \frac{EA}{EB} = \frac{4}{3}$ $\therefore$ EC = 4K , EB = 3K where $K \neq 0$

$\because \overline{AB} \cap \overline{CD} = \{E\}$ $\therefore EA \times EB = EC \times ED$ (well known problem)

then : $4K \times 3K = 9 \times 4$

$12K^2 = 36$

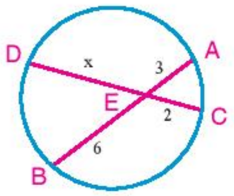
$K^2 = 3$

$K = \sqrt{3}$, $EB = 3\sqrt{3}$ cm

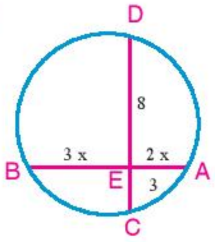
Try to solve

1 Find the value of x in each of the following figures (lengths are measured in centimetres)

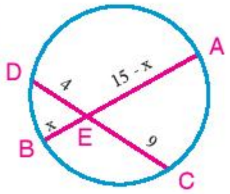
A



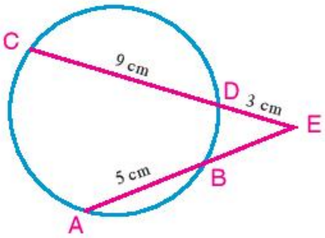
B



C



Example



2 **In the figure opposite:** $\overrightarrow{AB} \cap \overrightarrow{CD} = \{E\}$, A B = 5cm, CD = 9cm, ED = 3cm. Find the length of $\overline{BE}$

Solution

let B E = x cm.

$\because \overrightarrow{AB} \cap \overrightarrow{CD} = \{E\}$

$\therefore EB \times EA = ED \times EC$ (well known problem)

then: $x(x + 5) = 3(3 + 9)$

$x^2 + 5x - 36 = 0$

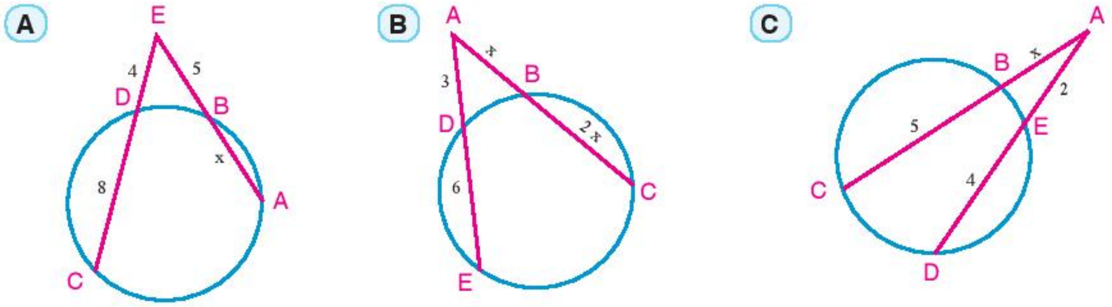
$(x - 4)(x + 9) = 0$

$\therefore x = 4$, $x = -9$ refused

$\therefore$ the length of $\overline{BE} = 4$cm.

Try to solve

2 Find the value of x in each of the following figures(lengths are measured in centimetres)

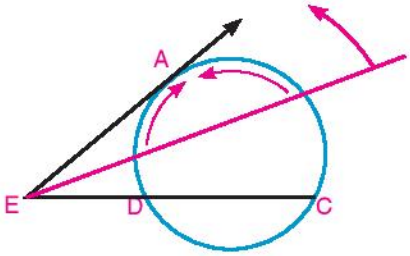


Corollary 1 **If E is a point outside the circle, $\overrightarrow{EA}$ is a tangent to the circle at A, $\overrightarrow{EC}$ cuts the circle at D and C, the $(EA)^2 = EC \times ED$**

In the figure opposite: $\overrightarrow{EA}$ is a tangent to the circle

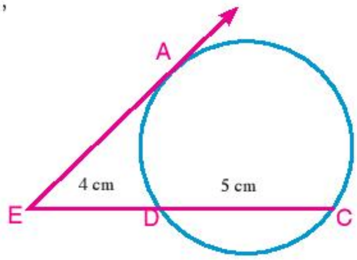
$\overrightarrow{EC}$ intersects the circle at D, C

$\therefore (EA)^2 = ED \times EC$



Example

3 **In the figure opposite**: $\overrightarrow{EA}$ is a tangent to the circle, $\overrightarrow{EA}$ intersects the circle at D and C respectively. where ED = 4cm, CD = 5cm, find the length of $\overline{EA}$



Solution

$\because \overrightarrow{EA}$ is tangent, $\overrightarrow{EC}$ is a a secant

$\therefore (EA)^2 = ED \times EC$ (Corollary)

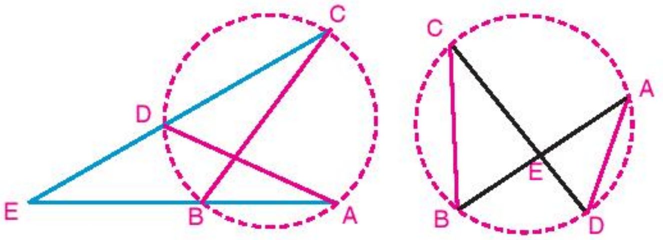
$(EA)^2 = 4(4 + 5) = 36$

$\therefore EA = 6$cm

Converse of the well known problem

If the two lines containing the two segments $\overline{AB}$ and $\overline{CD}$ intersect at a point E (A, B, C, D, and E are distinct points)
and EA × EB = EC × ED then: the points A, B, C and D lie on a circle.

Notice that:



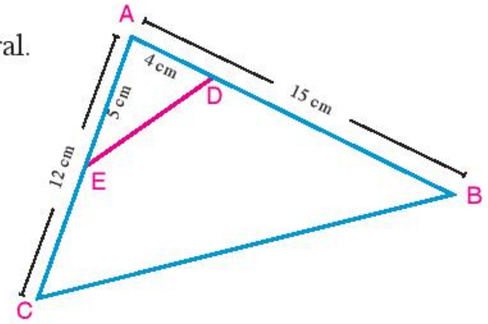
EA × EB = EC × ED

then $\frac{EA}{EC} = \frac{ED}{EB}$

- Is △ EAD ~ △ ECB? Why?
- Is m (∠A) = m (∠C)? Why?
- Do the points A, D, B and C lie on a circle? Explain your answer.

Example

4 ABC is a triangle in which AB = 15cm, AC = 12cm. D ∈ $\overline{AB}$ where A D = 4cm, E ∈ $\overline{AC}$ where AC = 5cm.
Prove that the figure DBCE is a cyclic quadrilateral.



Solution

∵ AD × AB = 4 × 15 = 60,
AE × AC = 5 × 12 = 60

∴ AD × AB = AE × AC

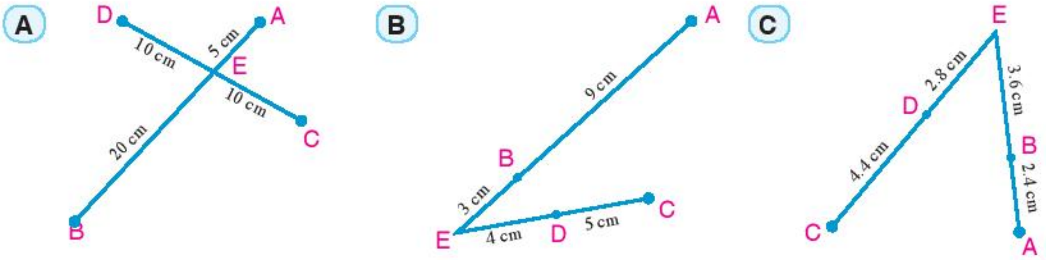
∵ $\overrightarrow{BE} \cap \overrightarrow{CE}$ = {A}, AD × AB = AE × AC

∴ the points D, B, C and E lie on a circle (converse of the well known problem)

then the figure DBCE is a cyclic quadrilateral

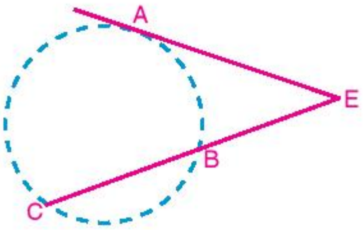
Try to solve

3 In which of the following figures, the points A, B, C and D lie on a circle? Explain your answer.



Corollary 2

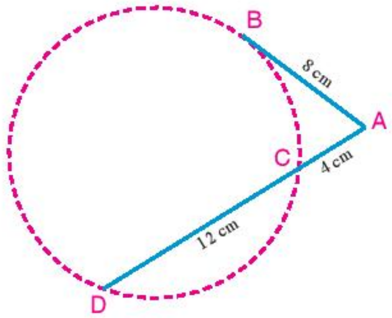
If $(EA)^2$ = EB × EC then $\overline{EA}$ is a tangent segment to the circle which passes through the points A, B and C.



Example

5 ABC is a triangle in which AB = 8cm, AC = 4cm, D ∈ $\overrightarrow{AC}$, D ∉ $\overline{AC}$ where CD = 12cm. prove that $\overline{AB}$ touches the circle which passes through the points B, C, and D

Solution



$\because AC \times AD = 4\,(4 + 12) = 64,$

$(AB)^2 = (8)^2 = 64$

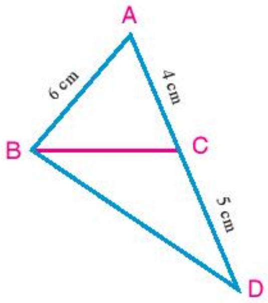
$\therefore (AB)^2 = AC \times AD$

$\therefore$ $\overline{AB}$ touches the circle which passes through the points B, C, and D at the point B.

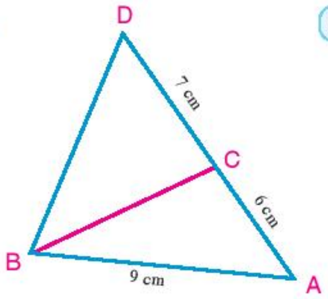
Try to solve

4 In which of the following figures is $\overline{AB}$ a tangent segment to the circle which passes through the points B, C, and D?

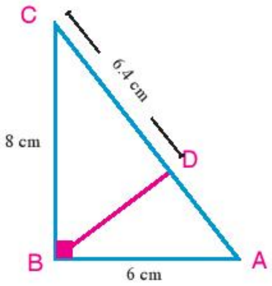
A



B

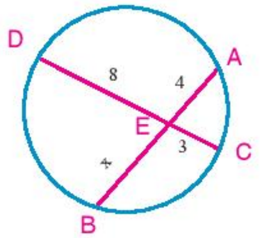


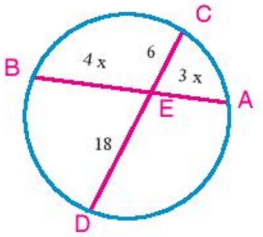
C

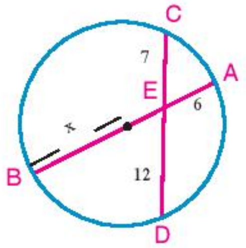


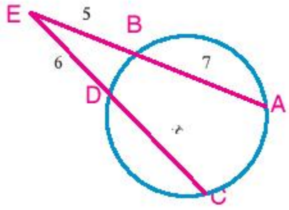
Exercises (2 - 4)

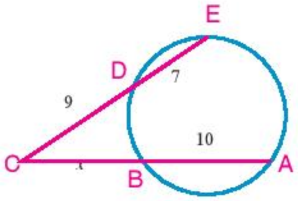
1 Use the calculator or mental math to find the numerical value of x in each of the following figures. (lengths are measured in centimetres)

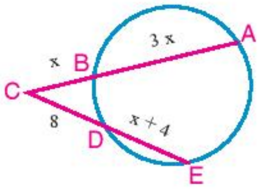
A


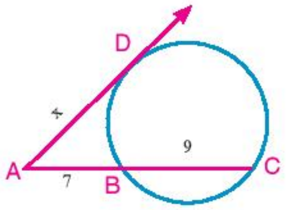
B


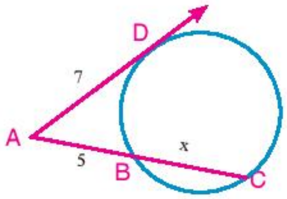
C


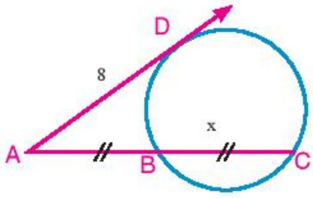
D


E


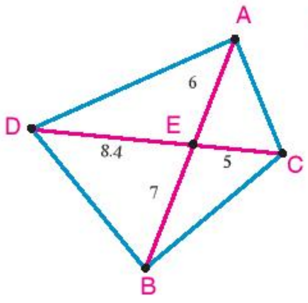
F


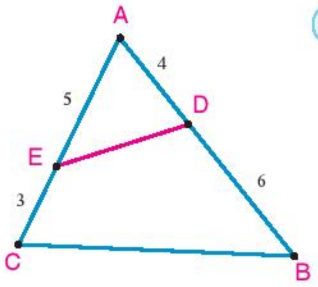
G


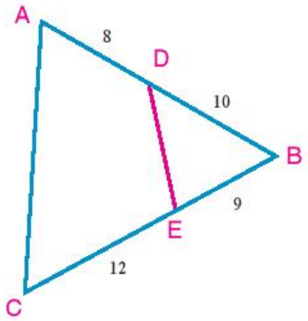
H


I


2 In which of the following figures, the points A, B, C and D Lie on a circle? Explain your answer. (the lengths are measured in centimetres)

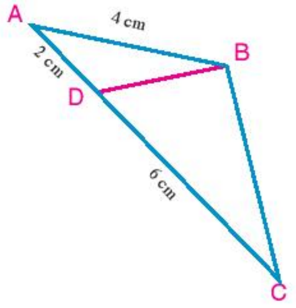
A


B


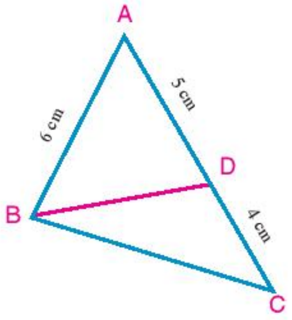
C


3 In which of the following figures, $\overline{AB}$ is a tangent to the circle passing through the points B, C and D.

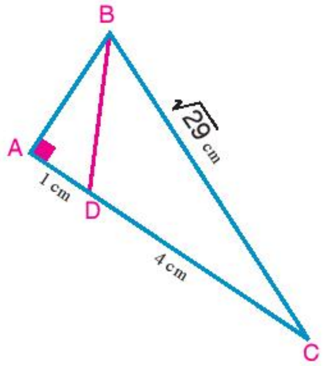
A



B

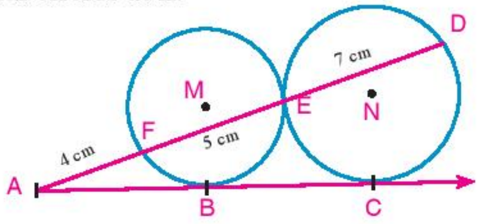


C



4 Two circles are intersected at A and B. $C \in \overleftrightarrow{AB}$ and $C \notin \overline{AB}$, From C, The two tangent segments $\overline{CX}$ and $\overline{CY}$ are drawn to the circle at X and Y respectively. Prove that CX = CY.

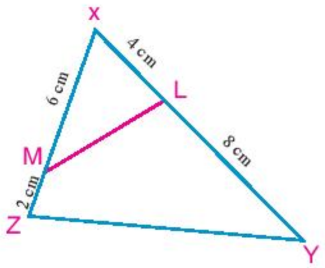
5 **In the figure opposite:** M and N are two tangential circles at E. $\overrightarrow{AC}$ touches the circle M at B, and touches the circle N at C, $\overrightarrow{AE}$ intersects the two circles at F and D respectively,
where AF = 4cm, FE = 5cm, ED = 7cm.
Prove that B is the midpoint of $\overline{AC}$



6 **In the figure opposite:** $L \ni \overline{XY}$ where XL = 4cm, YL = 8cm , $M \ni \overline{XZ}$ where XM = 6cm , ZM = 2cm
Prove that:

A $\triangle XLM \sim \triangle XZY$

B LYZM is a cyclic quadrilateral.



7 $\overline{AB} \cap \overline{CE} = \{E\}$, $AE = \frac{5}{12} BE$, $DE = \frac{3}{5} EC$. If BE = 6cm and CE = 5cm. prove that the points A, B, C and D lie on one circle.

8 ABC is a triangle. $D \ni \overline{BC}$ where DB = 5cm and DC = 4cm. If AC = 6cm. Prove that:

A $\overline{AC}$ is a tangent segment to the circle passing through the points A, B and D.

B $\triangle ACD \sim \triangle BCA$

C Area of ($\triangle ABD$): area of ($\triangle ABC$) = 5 : 9

9 Two concentric circles at M, their radii are 12cm, 7cm, $\overline{AD}$ is a chord in the larger circle to intersect the smaller circle at B and C respectively. Prove that $AB \times BD = 95$

Temple of Hatshepsut in Luxor

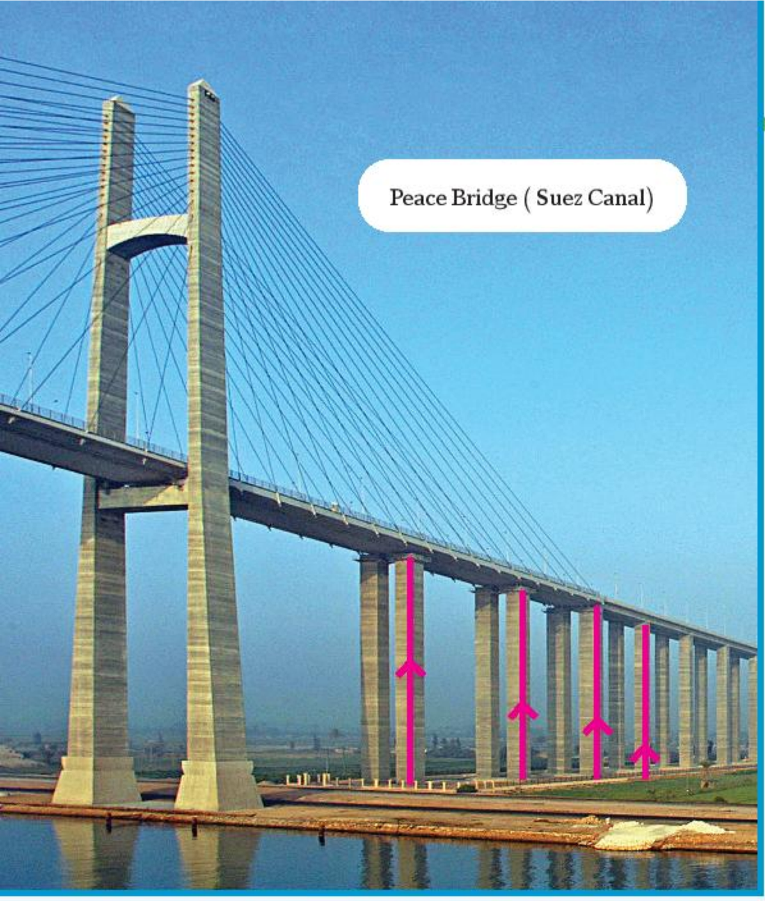
Unit objectives

by the end of the unit, the student should be able to:

- Recognize the theorem : if a line is drawn parallel to one side of a triangle and intersects the other two sides , then it divides them into segments whose lengths are proportional"
- Recognize TALIS general theorem" Given several coplanar parallel lines and two transversals..."
- Recognize the theorem" the bisector of the interior (or exterior) angle of a triangle at any vertex divides the opposite base.
- Find the power of a point w.r. to a circle (secants and tangents).
- Deduce the measures of angles resulting from the intersection of chords and the tangents in a circle .
- Solve applications about finding the length of each of the interior and the exterior bisectors .

Key - Terms

- Ratio
- Proportion
- Parallel
- Midpoint
- Median
- Transversal
- Bisector
- Interior Bisector
- Exterior Bisector
- Perpendicular



Lessons of the Unit

Lesson (3 - 1): Parallel Lines and Proportional Parts.

Lesson (3 - 2): Angle Bisectors and Proportional Parts.

Materials

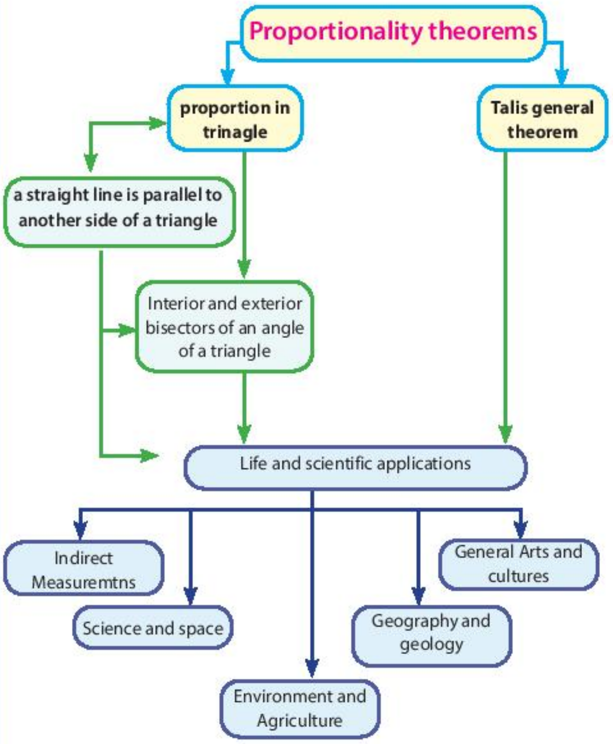
Geometric instruments for drawing and measurements – Computer – Graphic programs – datashow – squared paper – thread – scissors

Brief History

Mathematicians is an intellectual activity full of fun and makes the mind open and clear, and contributes to solving many problems and life challenges through representing or modeling it by relations in languge of mathematics and their symbols to be solved, then returned to the physical assets.

Ancient Eqyptions realized so , they set up temples and pyramids as straight lines, some are parallel and the other are transversals to them , and also plowed farmland in parallel straight lines, Greeks- has taken Geometry from ancient Egyptians, Euclid (300 BC) put geometric integrated system, was known as Euclidean Geometry depend on Five Axioms, the important one is parallelism Axiom which is " From a point does not belong to a straight line, it is possible to draw only a straight line passes through this point and parallel to the given line" The Euclidean Geometry dealing with plane figures (triangles - polygons-circles) and three dimensions figures, they also have practical applications in various fields including construction of roads, urban planning and preparation of maps, which rely on parallel lines and transversals to them according to the real length and drawing length (scale drawing).

Chart of the unit



3 - 1 Parallel Lines and Proportional Parts

You will learn

- Properties of the straight line which is parallel to any side of a triangle.
- Use proportion in calculation of lengths and in prove relations to line segments resulting from the transversals of parallel lines.
- Modelling and solving life problems including parallel lines and their transversals

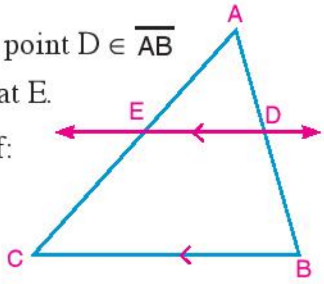
Key-terms

- Parallel
- Midpoint
- Median
- Transversal

Materials

- Geometric instruments
- Computer
- Graphic program
- Data show

Think and discuss



1- Draw the triangle A B C, determine the point $D \in \overline{AB}$ then draw $\overleftrightarrow{DE} \parallel \overline{BC}$ and intersects $\overline{AC}$ at E.

2- Find by measuring the length of each of: $\overline{AD}$, $\overline{DB}$, $\overline{AE}$ and $\overline{EC}$

3- Calculate each of the ratio $\frac{AD}{DB}$, $\frac{AE}{EC}$ and compare between them , what do you notice?

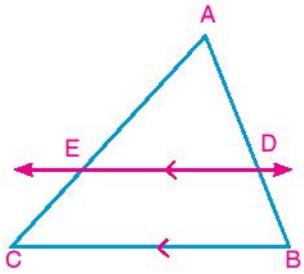
If the location of $\overleftrightarrow{DE}$ has been changed preserving parallelism to $\overline{BC}$.

Is the relation between $\frac{AD}{DB}$ and $\frac{AE}{EC}$ changed? What do you deduce?

Theorem 1

If a line is drawn parallel to one side of a triangle and intersects the other two sides, then it divides them into segments whose lengths are proportional.

(Proof is not required)



Given: ABC is a triangle, $\overleftrightarrow{DE} \parallel \overline{BC}$

R.t.p: $\frac{AD}{DB} = \frac{AE}{EC}$

Proof: $\because \overleftrightarrow{DE} \parallel \overline{BC}$

$\therefore \triangle ABC \sim \triangle ADE$ (similarity postulate)

then: $\frac{AB}{AD} = \frac{AC}{AE}$ (1)

$\because D \in \overline{AB}$, $E \in \overline{AC}$

$\therefore AB = AD + DB$, $AC = AE + EC$ (2)

from (1) and (2) we get:

$$\frac{AD + DB}{AD} = \frac{AE + EC}{AE}$$

then: $\frac{AD}{AD} + \frac{DB}{AD} = \frac{AE}{AE} + \frac{EC}{AE}$

$$1 + \frac{DB}{AD} = 1 + \frac{EC}{AE}$$

$$\therefore \frac{DB}{AD} = \frac{EC}{AE}$$

from the properties of proportion, we get: $\frac{AD}{DB} = \frac{AE}{EC}$ (Q.E.D)

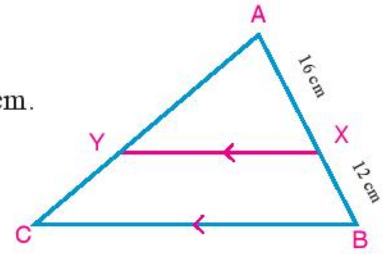
Notice that:

$\because \frac{AD}{DB} = \frac{AE}{EC}$ $\therefore \frac{AD + DB}{DB} = \frac{AE + EC}{EC}$

i.e.: $\frac{AB}{DB} = \frac{AC}{EC}$

Example

1 **In the figure opposite:** $\overline{XY} \parallel \overline{BC}$, A X = 16cm, BX = 12cm.



A If AY = 24cm, Find YC.

B If CY = 21cm, find AC.

Solution

A $\because \overline{XY} \parallel \overline{BC}$ $\therefore \frac{AX}{XB} = \frac{AY}{YC}$

and: $\frac{16}{12} = \frac{24}{YC}$ $\therefore YC = \frac{12 \times 24}{16} = 18$cm.

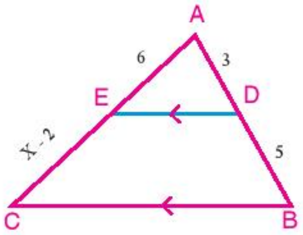
B $\because \overline{XY} \parallel \overline{BC}$ $\therefore \frac{AB}{BX} = \frac{AC}{CY}$

and: $\frac{16 + 12}{12} = \frac{AC}{21}$ $\therefore AC = \frac{28 \times 21}{12} = 49$cm.

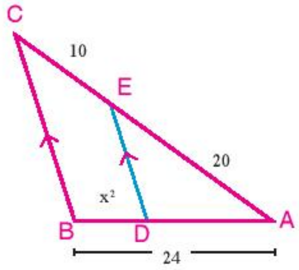
Try to solve

1 In each of the following figures : $\overline{DE} \parallel \overline{BC}$. Find the numerical value of x (lengths are measured in centimetres)

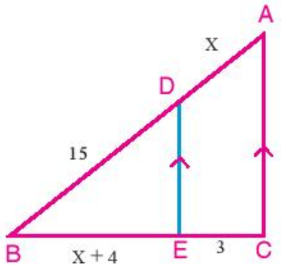
A



B



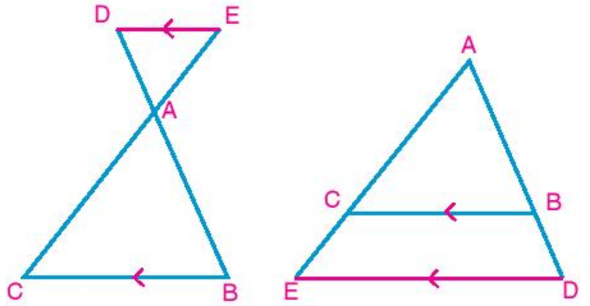
C



Corollary **If a straight line is drawn outside the triangle A B C parallel to one side of the triangle, say $\overline{BC}$ intersecting $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ at D and E, respectively as shown in the figures , then : $\frac{AB}{DB} = \frac{AC}{CE}$.**

and from the properties of the proportion we can deduce that:

$\frac{AD}{AB} = \frac{AE}{AC}$, $\frac{AD}{DB} = \frac{AE}{CE}$



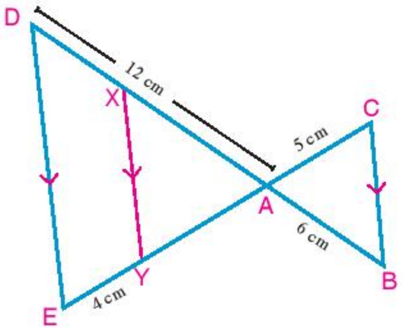
Example

2 **In the figure opposite:** $\overline{CE} \cap \overline{BD} = \{A\}$, $X \in \overline{AD}$

$Y \in \overline{AE}$ where $\overline{XY} // \overline{BC} // \overline{DE}$.

If AB = 6cm, AC = 5cm, AE = 12cm, EY = 4cm.

Find the length of each of $\overline{AE}$ and $\overline{DX}$.



Solution

$\because \overline{ED} // \overline{BC}$, $\overline{CE} \cap \overline{BD} = \{A\}$

$\therefore \frac{AD}{AB} = \frac{AE}{AC}$ and : $\frac{12}{6} = \frac{AE}{5}$ $\therefore AE = 10cm$

In $\triangle$ AED:

$\because \overline{XY} // \overline{ED}$ $\therefore \frac{AE}{EY} = \frac{AD}{DX}$

and $\frac{10}{4} = \frac{12}{DX}$ $\therefore DX = 4{,}8cm$

Try to solve

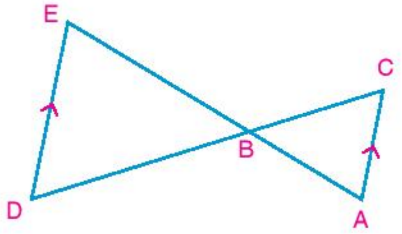
2 **In the figure opposite:** $\overline{DE} // \overline{AC}$, $\overline{AE} \cap \overline{CD} = \{B\}$

A If: AB = 8cm, BC = 9cm, BE = 12cm.

Find the length of $\overline{BD}$.

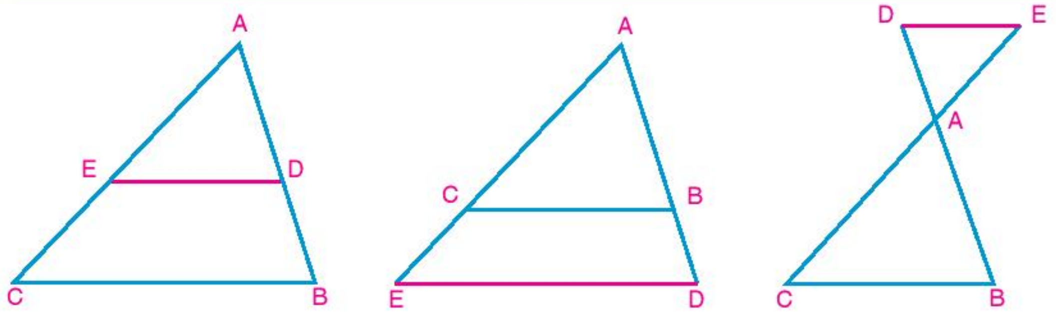
B If: AB = 6cm, BE = 9cm, CD = 18cm.

Find the length of $\overline{BC}$.



Converse of theorem 1

If a line intersects two sides of a triangle and divides them into segments whose lengths are proportional, then it is parallel to the third side of the triangle.



In the previous three figures: ABC is a triangle, $\overleftrightarrow{DE}$ intersects $\overleftrightarrow{AB}$ at D and $\overleftrightarrow{AC}$ at E,

$$\frac{AD}{DB} = \frac{AE}{EC} \text{ then } \overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$$

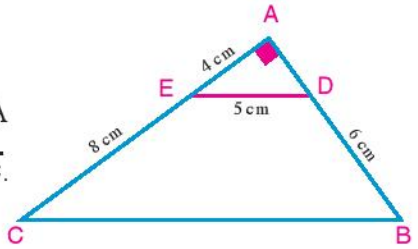
Logical thinking: Is $\triangle$ ADE $\sim \triangle$ ABC? Why? – Is $\angle$ ADE $\equiv \angle$ B? Explain your answer.

Write a proof for the converse of the theorem.

Example

3 **In the figure opposite:** ABC is a right angled triangle at A

A Prove that: $\overline{DE} \parallel \overline{BC}$. **B** Find the length of $\overline{BC}$.



Solution

A $\because$ the trinagle ADE is a right angled triangle at A

$\therefore AD = \sqrt{25 - 16} = 3\text{cm}$ (Pythagorian theorem)

$\because \frac{AD}{DB} = \frac{3}{6} = \frac{1}{2}$, $\frac{AE}{EC} = \frac{4}{8} = \frac{1}{2}$

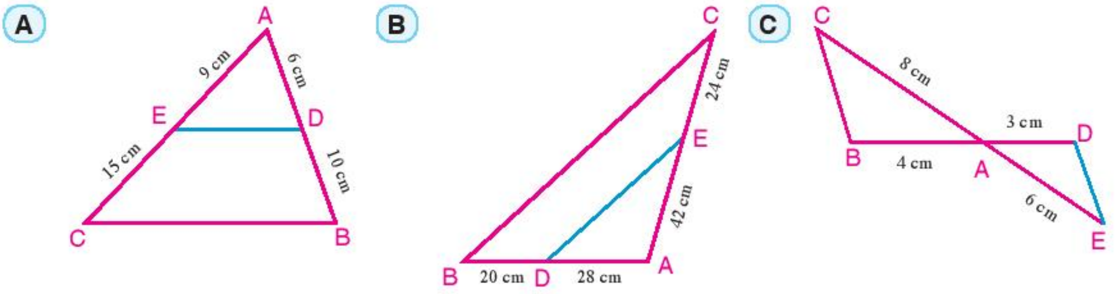
$\therefore \frac{AD}{DB} = \frac{AE}{EC}$ then $\overline{ED} \parallel \overline{BC}$.

B $\because \triangle ADE \sim \triangle ABC$ (why?) $\therefore \frac{AD}{AB} = \frac{DE}{BC} = \frac{1}{3}$,

then $\frac{5}{BC} = \frac{1}{3}$ $\therefore BC = 15\text{cm}$

Try to solve

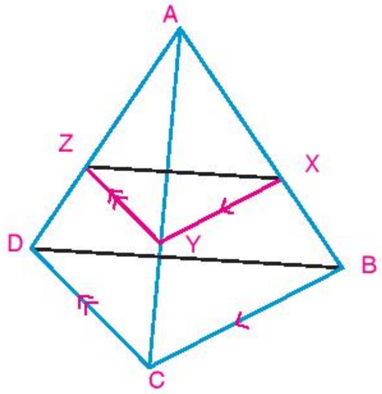
3 In each of the following figures, determine whether if $\overline{DE} \parallel \overline{BC}$ or not?



Example

4 ABCD is a quadrilateral in which $X \in \overline{AB}$, $Y \in \overline{AC}$ where $\overline{XY} \parallel \overline{BC}$, draw $\overline{YZ} \parallel \overline{CD}$ and intersects $\overline{AD}$ at Z. Prove that $\overline{XZ} \parallel \overline{DB}$.

Solution



In $\triangle ABC$:

$\because \overline{XY} \parallel \overline{BC}$ $\qquad \therefore \frac{AX}{XB} = \frac{AY}{YC}$ $\qquad$ (1)

In $\triangle ADC$:

$\because \overline{YZ} \parallel \overline{CD}$ $\qquad \therefore \frac{AZ}{ZD} = \frac{AY}{YC}$ $\qquad$ (2)

from (1) and (2) we deduce that: $\frac{AX}{XB} = \frac{AZ}{DZ}$

In $\triangle ABD$:

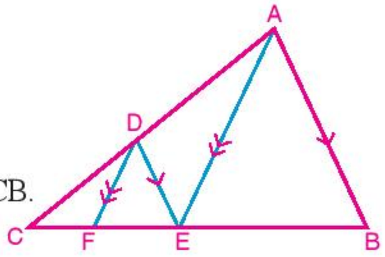
$\because \frac{AX}{XB} = \frac{AZ}{ZD}$ $\qquad \therefore \overline{XZ} \parallel \overline{DB}$

Try to solve

4 **In the figure opposite:** ABC is a triangle, $D \in \overline{AC}$,

$\overline{DE} \parallel \overline{AB}$, $\overline{DF} \parallel \overline{AE}$

Draw a chart which show how to prove that $(CE)^2 = CF \times CB$.

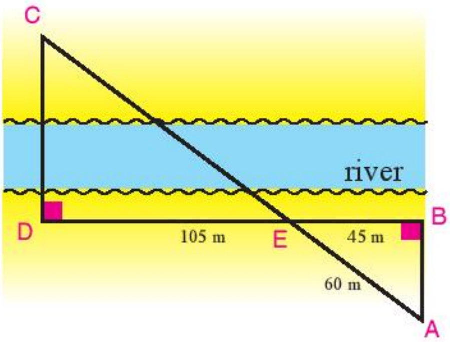


Example

5 GPS: to determine the location C,

Surveyors measure and prepare the scheme opposite.

Find the distance between the location C and the location A.



Solution

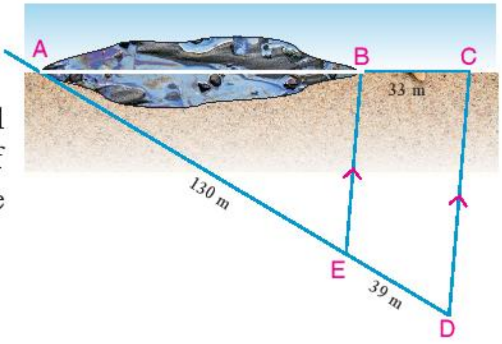
$\overline{AB} \perp \overline{BD}$, $\overline{CD} \perp \overline{BD}$ $\therefore \overline{AB} // \overline{CD}$

$\because$ $\overline{AC} \cap \overline{BD} = \{E\}$, $\overline{AB} // \overline{CD}$

$\therefore \frac{EA}{AC} = \frac{EB}{BD}$ then $\frac{60}{AC} = \frac{45}{45 + 105}$

$\therefore AC = \frac{60 \times 150}{45} = 200$ metres.

Try to solve



5 **Pollution Control:** A team of pollution control determined the location of an oil spot on one of the beaches as in the figure opposite. Calclate the length of the oil spot.

Think and discuss

You have noticed the possibility of using a parallel line to one of the sides of a triangle in many life applications.

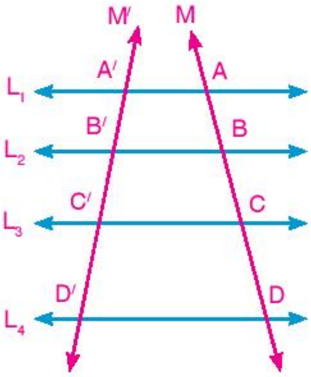
The figure opposite shows a door of one of the plant nurseries made from wooden parallel pieces and other transversals to them.

Is there a relation between the lengths of intercepted parts of these parallel pieces?

Modeling

To investigate the existence of a relation or not, make a model to the problem (make a mathematical model to the problem) as follows:

1- Draw the lines $L_1 // L_2 // L_3 // L_4$ and M, M$'$ are two transversals to them at A, B, C, D and A$'$, B$'$, C$'$, D$'$ as in the figure opposite.



2- Measure the lengths of line segments, and compare the following ratios:

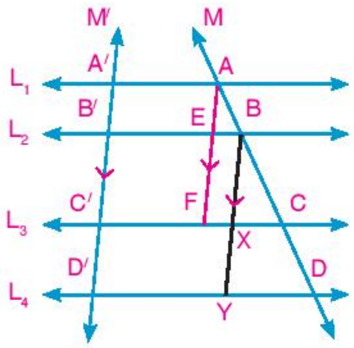
$\frac{AB}{A'B'}$, $\frac{BC}{B'C'}$, $\frac{CD}{C'D'}$, $\frac{AC}{A'C'}$.

What do you deduce?

Talis' Theorem

theorem 2 **Given several coplanar parallel lines and two transversals, then the lengths of the corresponding segments on the transversals are proportional.**

(Proof is not required)



Given: $L_1 // L_2 // L_3 // L_4$ and M, M' are two transversals to them

R.t.p : AB : BC : CD = A'B' : B'C' : C'D'

Proof : Draw $\overrightarrow{AF}$ // M', and intersects L_2 at E, and L_3 at F,

$\overrightarrow{BY}$ // M' and intersects L_3 at X and L_4 at Y.

$\because \overline{AA'} // \overline{EB'}$, $\overline{AE} // \overline{A'B'}$

$\therefore$ AEB'A' is a paralleogram , then AE = A'B'

Similarly: EF = B'C' , BX = B' C' , XY = C'D'

In $\triangle$ACF:

$\because \overline{BE} // \overline{CF}$ $\qquad \therefore \frac{AB}{BC} = \frac{AE}{EF}$

then: $\frac{AB}{BC} = \frac{A'B'}{B'C'}$, $\frac{AB}{A'B'} = \frac{BC}{B'C'}$ (exchange the means) (1)

similarly $\triangle$BDY:

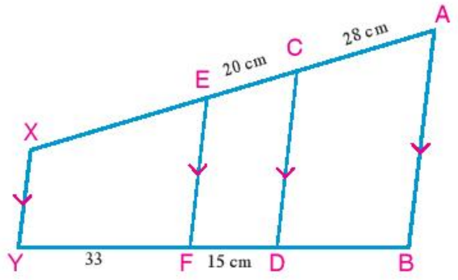
$\therefore \frac{BC}{CD} = \frac{B'C'}{C'D}$, $\frac{BC}{B'C'} = \frac{CD}{C'D'}$ (exchange the means) (2)

from (1) and (2) we get:

$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$

$\therefore$ A B : B C : C D = A' B' : B' C' : C' D' Q.E.D.

Example



6 **In the figure opposite:** $\overline{AB} // \overline{CD} // \overline{EF} // \overline{XY}$,
AC = 28cm, CE = 20cm, DF = 15cm, FY = 33cm.
Find the length of each of: $\overline{BD}$ and $\overline{EX}$

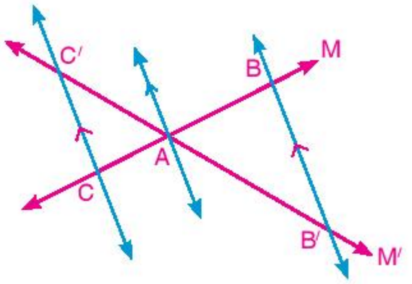
Solution

$\because \overline{AB} // \overline{CD} // \overline{EF} // \overline{XY}$

$\therefore \frac{AC}{BD} = \frac{CE}{DF} = \frac{EX}{FY}$

$\frac{28}{BD} = \frac{20}{15} = \frac{EX}{33}$ $\qquad \therefore$ BD = 21cm , EX = 44cm.

Special cases



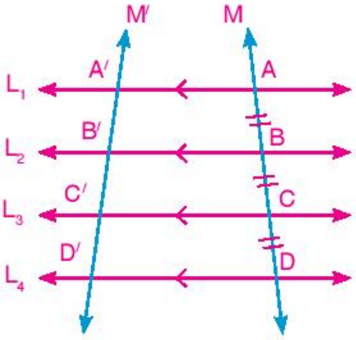
1- If the two lines M , $M^{/}$ intersect at the point A

and: $\overleftrightarrow{BB^{/}} // \overleftrightarrow{CC^{/}}$, then: $\frac{AB}{AC} = \frac{AB^{/}}{AC^{/}}$

and conversely: If: $\frac{AB}{AC} = \frac{AB^{/}}{AC^{/}}$

then: $\overleftrightarrow{BB^{/}} // \overleftrightarrow{CC^{/}}$

Talis's Special theorem

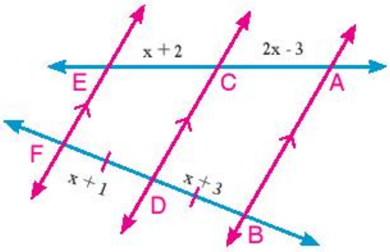


2- If the lengths of the segments on the transversals are equal, then the lengths of the segments on any other transversal will be also equal.

In the figure opposite $L_1 // L_2 // L_3 // L_4$ and M, $M^{/}$ are two transversals to them and AB = BC = CD then: $A^{/}B^{/} = B^{/}C^{/} = C^{/}D^{/}$

Example

7 **In te figure opposite , find the numerical value of x and y**.



Solution

$\because \overline{AB} // \overline{CD} // \overline{EF}$, BD = DF

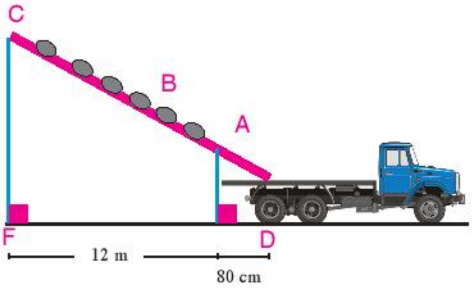
$\therefore$ AC = CE

then: $2x - 3 = x + 2$ $\quad \therefore x = 5$

$\because$ BD = DF , x = 5 $\quad \therefore y + 3 = 5 + 1$ $\quad \therefore y = 3$

Example

8 **Industry:** Fertilizer packages produced from one of the factories are transfered by sliding on a tube that is inclined and carried on to trucks to the centre of distributions as in the figure opposite.



If D, E and F are the projections of the points A, B and C on the horizontal respectively, AB = 1.2m, DE = 80cm , EF = 12m. Find the length of the tube to the nearest metre .

Solution

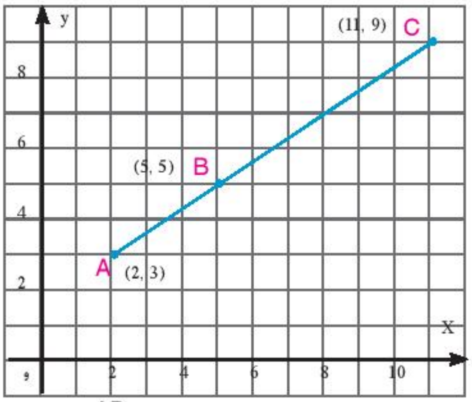
$\because$ D, E and F are projections to the points A, B and C on the horizontal $\therefore \overline{AD} // \overline{BE} // \overline{CF}$

$\because \overline{AD} // \overline{BE} // \overline{CF}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{DF}$ are two transversals to them $\quad \therefore \frac{AC}{AB} = \frac{DF}{DE}$: $\frac{AC}{1.2} = \frac{12 + 0.8}{0.8}$

$\therefore AC = \frac{1.2 \times 12.8}{0.8} = 19.2$ m $\quad \therefore AC \simeq 19$ metres

Try to solve

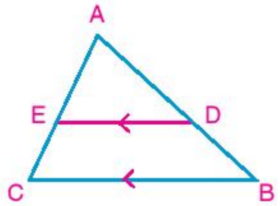
6 **Logical thinking**



From the figure , find the value of $\frac{AB}{BC}$ in different methods, if possible.
Did you get the same result ?

Exercises (3 - 1)

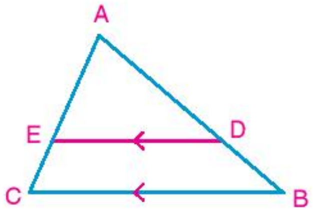
1 **In the figure opposite:** $\overline{ED}$ // $\overline{BC}$. Complete:



A if $\frac{AD}{DB} = \frac{5}{3}$ then : $\frac{AB}{BD} = \frac{\dots\dots}{\dots\dots}$ and $\frac{CE}{ED} = \frac{\dots\dots}{\dots\dots}$

B if $\frac{AE}{AC} = \frac{4}{7}$, then : $\frac{CE}{EA} = \frac{\dots\dots}{\dots\dots}$ and $\frac{BD}{AB} = \frac{\dots\dots}{\dots\dots}$

2 **In the figure opposite:** $\overline{DE}$ // $\overline{BC}$. Determine the correct statements in each of the following:



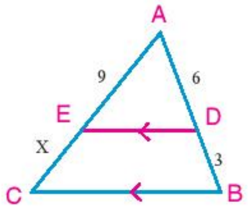
A $\frac{AB}{DB} = \frac{AE}{EC}$ B $\frac{AD}{AE} = \frac{BD}{EC}$

C $\frac{AB}{BD} = \frac{AC}{AE}$ D $\frac{AB}{BD} = \frac{AC}{CE}$

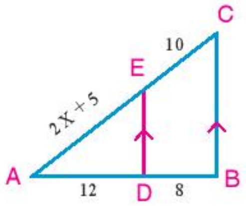
E $\frac{AC}{AD} = \frac{AB}{AE}$ F $\frac{CE}{BD} = \frac{AC}{AB}$

3 **In each of the following figures:** $\overline{DE}$ // $\overline{BC}$. Find the numerical value of x (length in centimetres).

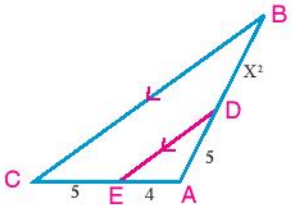
A



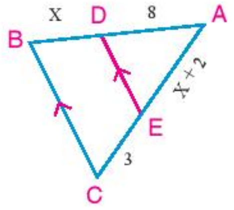
B



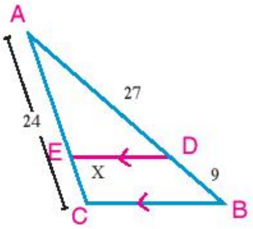
C



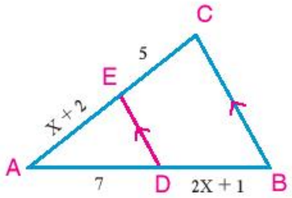
D



E



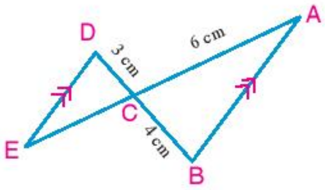
F



4 **In the figure opposite:** $\overline{AB}$ // $\overline{DE}$ and $\overline{AE} \cap \overline{BD} = \{C\}$

AC = 6cm, BC = 4cm and CD = 3cm.

Find the length $\overline{AE}$

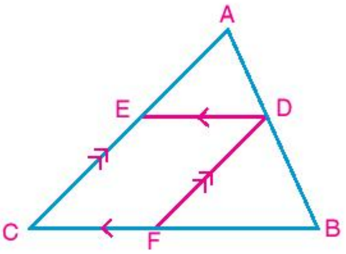


5 $\overline{XY} \cap \overline{ZL} = \{M\}$, where $\overline{XZ}$ // $\overline{LY}$. If XM = 9cm, Y M = 15cm and ZL = 36 cm, find the length of $\overline{ZM}$.

6 For each of the following, use the figure opposite and the given data to find the value of x:

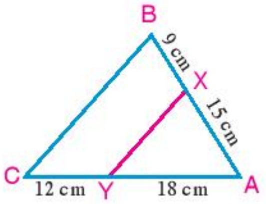
A AD = 4 , BD = 8 , CE = 6 and AE = x.

B AE = x , EC = 5 , AD = x – 2 and AD = 3.

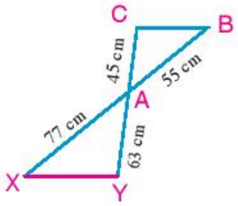


7 In each of the following figures, Is $\overline{XY}$ // $\overline{BC}$?

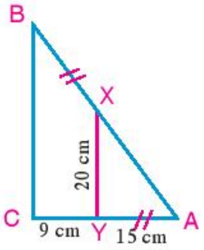
A



B



C



8 XYZ is a triangle in which XY= 14cm, XZ = 21cm and L ∋ $\overline{XY}$ where XL = 5,6cm and M ∋ $\overline{XZ}$ where XM = 8.4cm. Prove that $\overline{LM}$ // $\overline{YZ}$

9 In the triangle ABC, D ∋ $\overline{AB}$, E $\overline{AC}$,∋and 5AE = 4 EC.
If AD = 10 cm and DB = 8cm. Is $\overline{DE}$ // $\overline{BC}$? Explain your answer.

10 ABCD is a quadrilateral, its diagonals are intersected at E. If AE = 6cm, BE = 13cm, EF = 10cm and EC = 7.8 cm. prove that ABCD is a trapezium.

3 - 2 Angle Bisectors and Proportional Parts

You will learn

- Properties of bisectors of angles of triangles .
- Use proportion to calculate the lengths of line segments resulting from bisecting an angle in a triangle.
- Modelling and solving life problems including bisectors of angles of triangle

Key-terms

- Bisector
- Interior Bisector
- Exterior Bisector
- Perpendicular

Materials

- Geometric instruments for drawing.
- Computer and Graph programs.
- Data show.

Group work

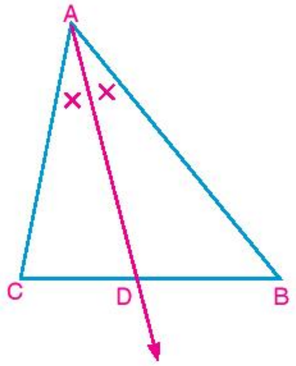
1- Draw the triangle ABC, and draw $\overrightarrow{AD}$ to intersect $\overline{BC}$ at D.

2- Measure each of $\overline{BD}$, $\overline{CD}$, $\overline{AB}$, $\overline{AC}$.

3- Calculate each of the two ratios $\frac{BD}{DC}$, $\frac{BA}{AC}$ and compare between them.

What do you deduce?

4- Repeat the previous work many times. Does your deduction verify? Express your deduction.

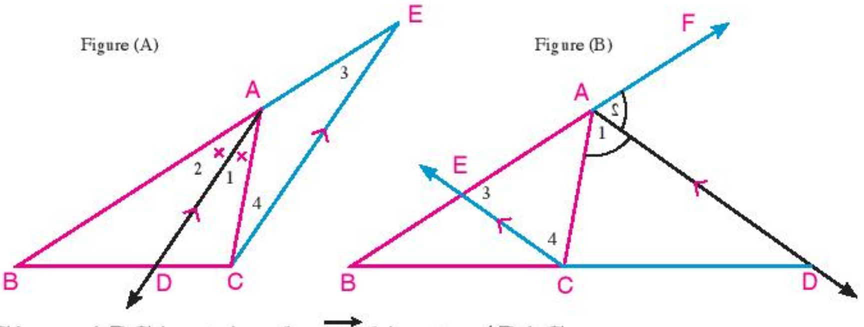


Bisector of an Angle of a Triangle

theorem 3

The bisector of the interior or exterior angle of a triangle at any vertex divides the opposite base of the triangle internally or externally into two parts the ratio of their lengths is equal to ratio of the lengths of the other two sides of the triangle.

(Proof is not required)



Given: ABC is a triangle, $\overrightarrow{AD}$ bisects $\angle BAC$ (internally in figure A , externally in figure B).

R.t.p. : $\frac{BD}{DC} = \frac{AB}{AC}$

Proof : Draw $\overleftrightarrow{CE}$ // $\overleftrightarrow{AD}$ and intersects $\overrightarrow{BA}$ at E. Follow the following chart and write the proof.

$\overrightarrow{AD}$ bisects $\angle A$ → $\angle 1 \equiv \angle 2$

$\overline{CE} \parallel \overrightarrow{AD}$ → $\angle 1 \equiv \angle 4$ (alternate), $\angle 3 \equiv \angle 2$ (corresponding), $\frac{BD}{DC} = \frac{AB}{AC}$

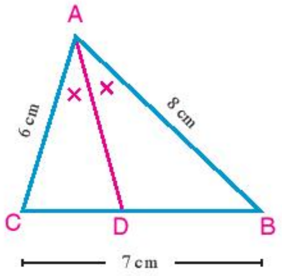
$\angle 1 \equiv \angle 2$, $\angle 1 \equiv \angle 4$ → $\angle 3 \equiv \angle 4$ → $\overline{AE} \equiv \overline{AC}$

$\frac{BD}{DC} = \frac{AB}{AE}$, $\overline{AE} \equiv \overline{AC}$ → $\frac{BD}{DC} = \frac{AB}{AC}$

Example

1 ABC is a triangle in which A B = 8cm , AC = 6cm , BC = 7cm , $\overrightarrow{AD}$ bisects $\angle BAC$ and intersects $\overline{BC}$ at D. Find the length of $\overline{DB}$, $\overline{DC}$

Solution



$\because \overrightarrow{AD}$ bisects $\angle BAC$ $\quad \therefore \frac{DB}{DC} = \frac{AB}{AC}$ (theorem)

$\because AB = 8cm, AC = 6cm$ $\quad \therefore \frac{BD}{DC} = \frac{8}{6} = \frac{4}{3}$

$\because BC = BD + DC = 7cm$ $\quad \therefore \frac{DB}{7 - BD} = \frac{4}{3}$

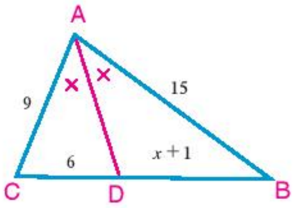
$3BD = 28 - 4\,BD$ (cross multiplication)

$7BD = 28$ $\quad \therefore BD = 4cm$, $CD = 3cm$

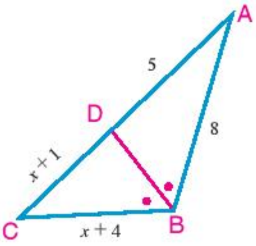
Try to solve

1 In each of the following figures, find the numerical value of x (lengths are measured in centimetres)

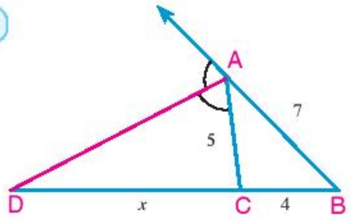
A



B



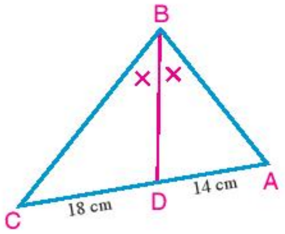
C



Example

2 ABC is a triangle. Draw $\overrightarrow{DB}$ bisects $\angle B$, intersects $\overline{AC}$ at D, where AD = 14cm, DC = 18cm. If the perimeter of $\triangle ABC$ equals 80cm, find the length of each of: $\overline{BC}$, $\overline{AC}$.

Solution



In $\triangle ABC$

$\because \overrightarrow{DB}$ bisects $\angle B$ $\quad \therefore \frac{AB}{BC} = \frac{AD}{DC}$

$\therefore \frac{AB}{BC} = \frac{14}{18} = \frac{7}{9}$

$\because$ the perimeter of $\triangle ABC = 80cm$, $AC = 14 + 18 = 32cm$

$\therefore AB + BC = 80 - 32 = 48cm$

$\because \frac{AB}{BC} = \frac{7}{9}$ $\therefore \frac{AB + BC}{BC} = \frac{7 + 9}{9}$ (properties of proportion)

then $\frac{48}{BC} = \frac{16}{9}$ $\therefore BC = 27cm$, $AB = 21cm$

Try to solve

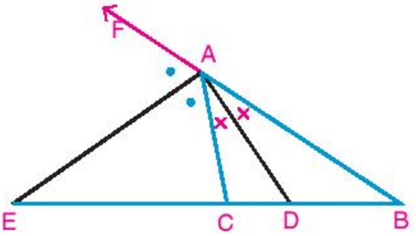
2 ABC is a right angled triangle at B. draw $\overrightarrow{AD}$ bisects $\angle A$, and intersects $\overline{BC}$ at D. If the length of $\overline{BD}$ equals 24cm, BA : AC = 3 : 5, find the perimeter of $\triangle$ ABC.

Important note

1- In the triangle ABC where $AB \neq AC$:

If $\overrightarrow{AD}$ bisects $\angle BAC$,

$\overrightarrow{AE}$ bisects the exterior angle of the triangle at A.



then: $\frac{DB}{DC} = \frac{AB}{AC}$, $\frac{BE}{EC} = \frac{AB}{AC}$

then $\frac{DB}{DC} = \frac{BE}{EC}$

i.e. $\overline{BC}$ is divided internally at D and externally at E by the same ratio

then the two bisectors $\overrightarrow{AD}$ and $\overrightarrow{AE}$ are perpendicular. why?

2- If AB > AC and the bisector of $\angle A$ intersects $\overline{BC}$ at D where BD > DC. The bisector of the exterior angle of the triangle at A intersects $\overleftrightarrow{BC}$ at E where BE > EC.

Critical thinking

- What happend to the point B, when AC is enlarged?
- When does the point D lie if AC = AB? and what is the position of $\overrightarrow{AE}$ w.r. to $\overline{BC}$ at then?
- What is the relation between DC and DB, when AC > AB? Where does the point E lie at then? Compare your answer with your classmate.

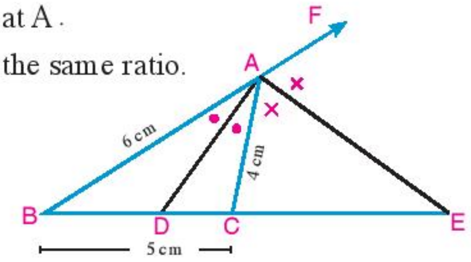
Example

3 ABC is atriangle in which AB = 6 cm, AC = 4 cm , BC = 5cm. $\overrightarrow{AD}$ bisects $\angle A$ and intersects $\overline{BC}$ at D, $\overrightarrow{AE}$ bisects the exterior angle at A and intersects $\overleftrightarrow{BC}$ at E. Calculate the length of $\overline{DE}$.

Solution

$\because \overrightarrow{AD}$ bisects $\angle A$ and $\overrightarrow{AE}$ bisects the exterior angle at A .

$\therefore$ D, and E divide $\overline{BC}$ internally and externally by the same ratio.



i.e : $\frac{BD}{DC} = \frac{BE}{EC} = \frac{BA}{AC}$

$\therefore \frac{BD}{DC} = \frac{BE}{EC} = \frac{6}{4} = \frac{3}{2}$

$\because BC = BD + DC = 5$, $BE - EC = BC = 5$

From the properties of proportion , we get

$\frac{BD + DC}{DC} = \frac{3+2}{2}$ $\qquad$ $\frac{5}{DC} = \frac{5}{2}$ $\quad \therefore DC = 2$

$\frac{BE - EC}{EC} = \frac{3-2}{2}$ $\qquad$ $\frac{5}{EC} = \frac{1}{2}$ $\quad \therefore EC = 10$

then $DE = DC + CE$ $\qquad$ $DE = 2 + 10 = 12cm$

Finding the length of the interior and the exterior bisectors of an angle of a triangle

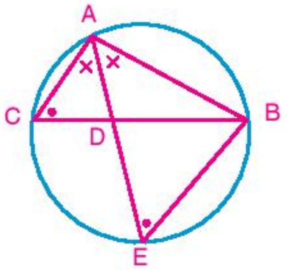
well known problem

If $\overrightarrow{AD}$ bisects $\angle A$ in $\triangle ABC$ internally and intersects $\overline{BC}$ at D then: $AD = \sqrt{AB \times AC - BD \times DC}$ *(Proof is not required)*

Given: ABC is a triangle, $\overrightarrow{AD}$ bisects $\angle$ BAC internally, $\overrightarrow{AD} \cap \overline{BC} = \{D\}$

R.t.p.: $(AD)^2 = AB \times AC - BD \times DC$

Proof: Draw a circle passes through $\triangle$ A B C and intersects $\overrightarrow{AD}$ at E, draw $\overline{BE}$



then: $\triangle ACD \sim \triangle AEB$ (why)?, $\frac{AD}{AB} = \frac{AC}{AE}$

$\therefore AD \times AE = AB \times AC$

$AD \times (AD + DE) = AB \times AC$

$(AD)^2 = AB \times AC - AD \times DE$

$(AD)^2 = AB \times AC - BD \times DC$

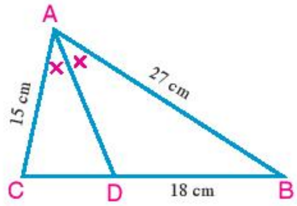
i.e.: $AD = \sqrt{AB \times AC - BD \times DC}$

Remember

$AD \times DE = BD \times DC$

Example

4 ABC is a triangle in which AB = 27cm, AC = 15cm. $\overrightarrow{AD}$ bisects $\angle$A and intersects $\overline{BC}$ at D. If BD = 18cm. Calculate the length of $\overline{AD}$.



Solution

$\because \overrightarrow{AD}$ bisects $\angle BAC$ $\qquad \therefore \frac{BD}{DC} = \frac{AB}{AC}$

then $\frac{18}{DC} = \frac{27}{15}$ $\qquad \therefore DC = 10cm$

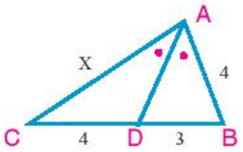
$\because AD = \sqrt{AB \times AC - BD \times DC}$

$\therefore AD = \sqrt{27 \times 15 - 18 \times 10} = \sqrt{225} = 15$ cm

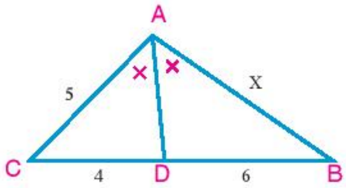
Try to solve

3 In each of the following figures (lengths are measured in centimetres). Calculate the value of x and the length of $\overline{AD}$

A



B



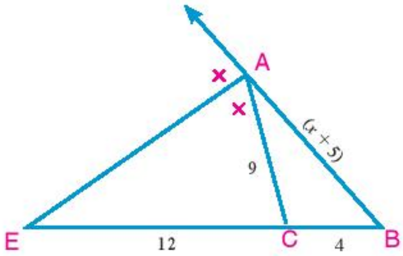
Notice that: In the figure opposite $\overrightarrow{AE}$ bisects $\angle$ BAC externally and intersects $\overrightarrow{BC}$ at E. then:

$AE = \sqrt{BE \times EC - AB \times AC}$

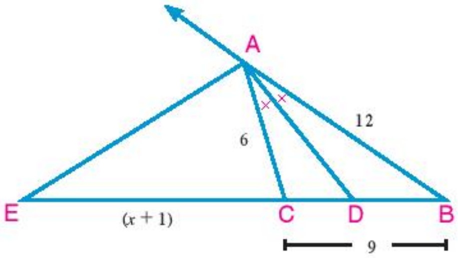
Try to solve

4 In each of the following figures (lengths are measured in centimetres). Calculate the value of x, and the lngth of $\overline{AE}$

A



B



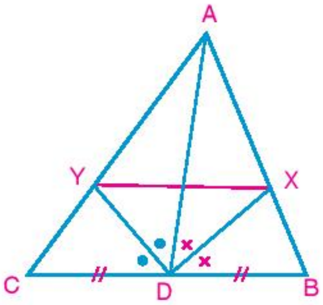
Example

5 **In the figure opposite:** $\overline{AD}$ is a median in $\triangle$ ABC

$\overrightarrow{DX}$ bisects $\angle$ADB. and intersects $\overline{AB}$ at X.

$\overrightarrow{DY}$ bisects $\angle$ADC and intersects $\overline{AC}$ at Y.

Prove that : $\overline{XY} // \overline{BC}$.



Solution

In $\triangle$ ADB: $\because \overrightarrow{DX}$ bisects $\angle$ADB $\quad \therefore \frac{AD}{DB} = \frac{AX}{XB}$ (1)

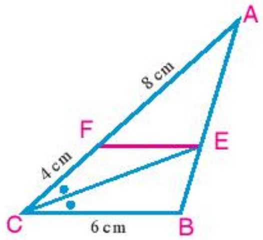
In $\triangle$ ADC: $\because \overrightarrow{DY}$ bisects $\angle$ADC $\quad \therefore \frac{AD}{DC} = \frac{AY}{YC}$ (2)

In $\triangle$ ABC: $\because \overline{AD}$ is a median $\quad \therefore$ DB = DC (3)

from (1), (2) and (3) $\frac{AX}{XB} = \frac{AY}{YC}$ then $\overline{XY} // \overline{BC}$.

Try to solve

5 In each of the following figures, prove that: $\overline{EF} \parallel \overline{BC}$

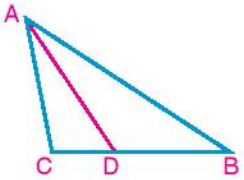


Special cases

1- In $\triangle$ ABC:

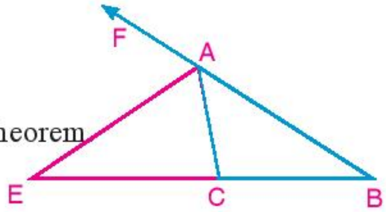
If D $\in \overline{BC}$, where $\frac{BD}{DC} = \frac{BA}{AC}$

then: $\overrightarrow{AD}$ bisects $\angle$ BAC



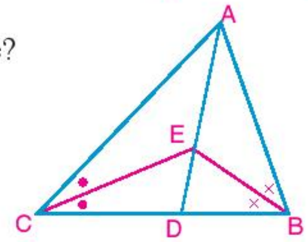
If E $\in \overleftrightarrow{BC}$, E $\notin \overline{BC}$, where $\frac{BE}{EC} = \frac{BA}{AC}$

then: $\overrightarrow{AE}$ bisects the exterior angle of $\triangle$ ABC at A , and it is defined by the converse of the previous theorem



2- In the figure opposite:

$\overrightarrow{BE}$, $\overrightarrow{CE}$ are bisectors of angles B and C intersecting at the point E $\in \overrightarrow{AD}$. What do you deduce?

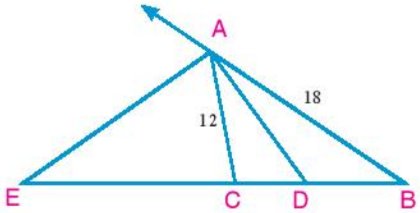


Fact: The bisectors of angles of a triangle are concurrent.

Example

6 ABC is a triangle in which AB = 18cm, BC = 15cm, AC = 12cm, D $\in \overline{BC}$, where BD = 9cm, $\overrightarrow{AE} \perp \overrightarrow{AD}$ and intersects $\overleftrightarrow{BC}$ at E. prove that $\overrightarrow{AD}$ bisects $\angle$BAC, then find the length of $\overline{CE}$.

Solution



In $\triangle$ ABC: $\frac{AB}{AC} = \frac{18}{12} = \frac{3}{2}$

CD = BC – BD = 15 – 9 = 6cm

$\therefore \frac{BD}{DC} = \frac{9}{6} = \frac{3}{2}$

$\because \frac{BD}{DC} = \frac{AB}{AC}$ $\qquad$ $\overrightarrow{AD}$ bisects $\angle$BAC

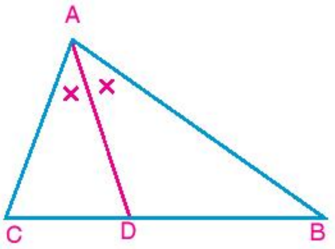
$\because \overrightarrow{AE} \perp \overrightarrow{AD}$ and intersects $\overleftrightarrow{BC}$ at E

$\therefore \overrightarrow{AE}$ bisects the exterior angle of $\triangle$ ABC at A $\qquad$ then $\frac{BE}{EC} = \frac{AB}{AC}$

$\because$ BE = BC + CE $\quad \therefore \frac{15 + CE}{CE} = \frac{18}{12}$, $\quad$ CE = 30cm

Exercises (3 - 2)

1 **In the figure opposite:** $\overrightarrow{AD}$ bisects $\angle A$. Complete:



A $\frac{BD}{DC} =$

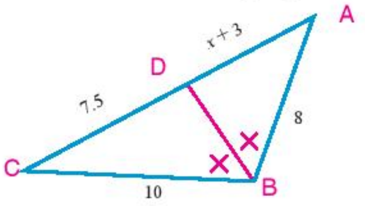
B $\frac{AC}{AB} =$ ....

C $\frac{BD}{BA} =$ ....

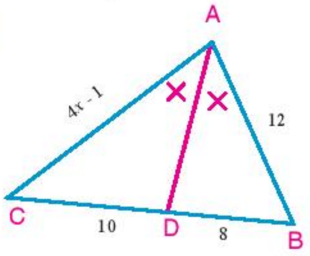
D $AB \times CD =$ ....

2 **In each of the following figures**: find the value of X (lengths are estimated in centimetres)

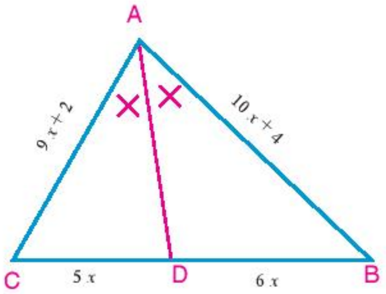
A



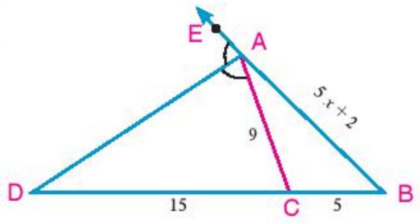
B



C



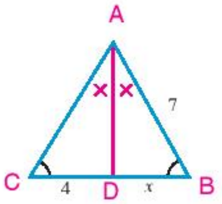
D



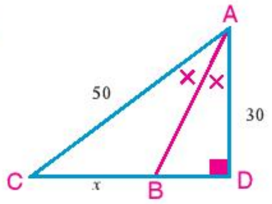
3 ABC is a triangle. its perimeter is 27cm. $\overrightarrow{BD}$ bisects $\angle$ B and intersects $\overline{AC}$ at D. If AD = 4cm and CD = 5cm, find the length of $\overline{AB}$, $\overline{BC}$ and $\overline{AD}$

4 **In each of the following figures**, find the value of x then find the perimeter of $\triangle ABC$.

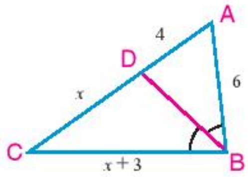
A



B



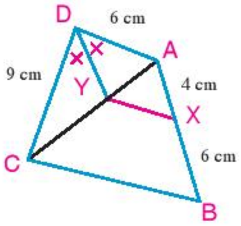
C



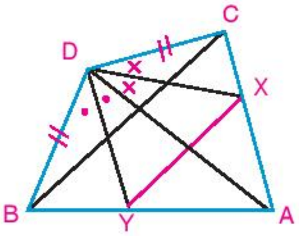
5 ABC is a triangle in which AB = 8cm, AC = 4cm and BC = 6cm and $\overrightarrow{AD}$ bisects $\angle A$ and intersects $\overline{BC}$ at D and $\overrightarrow{AE}$ bisects the exterior angle at A and intersects $\overleftrightarrow{BC}$ at E. Find the length of $\overline{DE}$, $\overline{AD}$ and $\overline{AE}$.

6 In each of the following figures, prove that $\overline{XY} \parallel \overline{BC}$

A

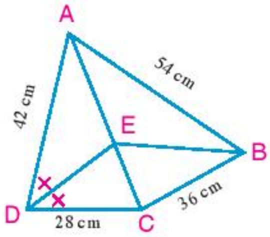


B

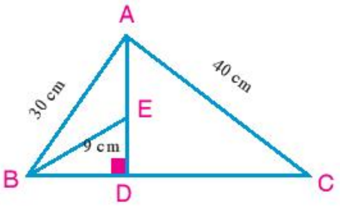


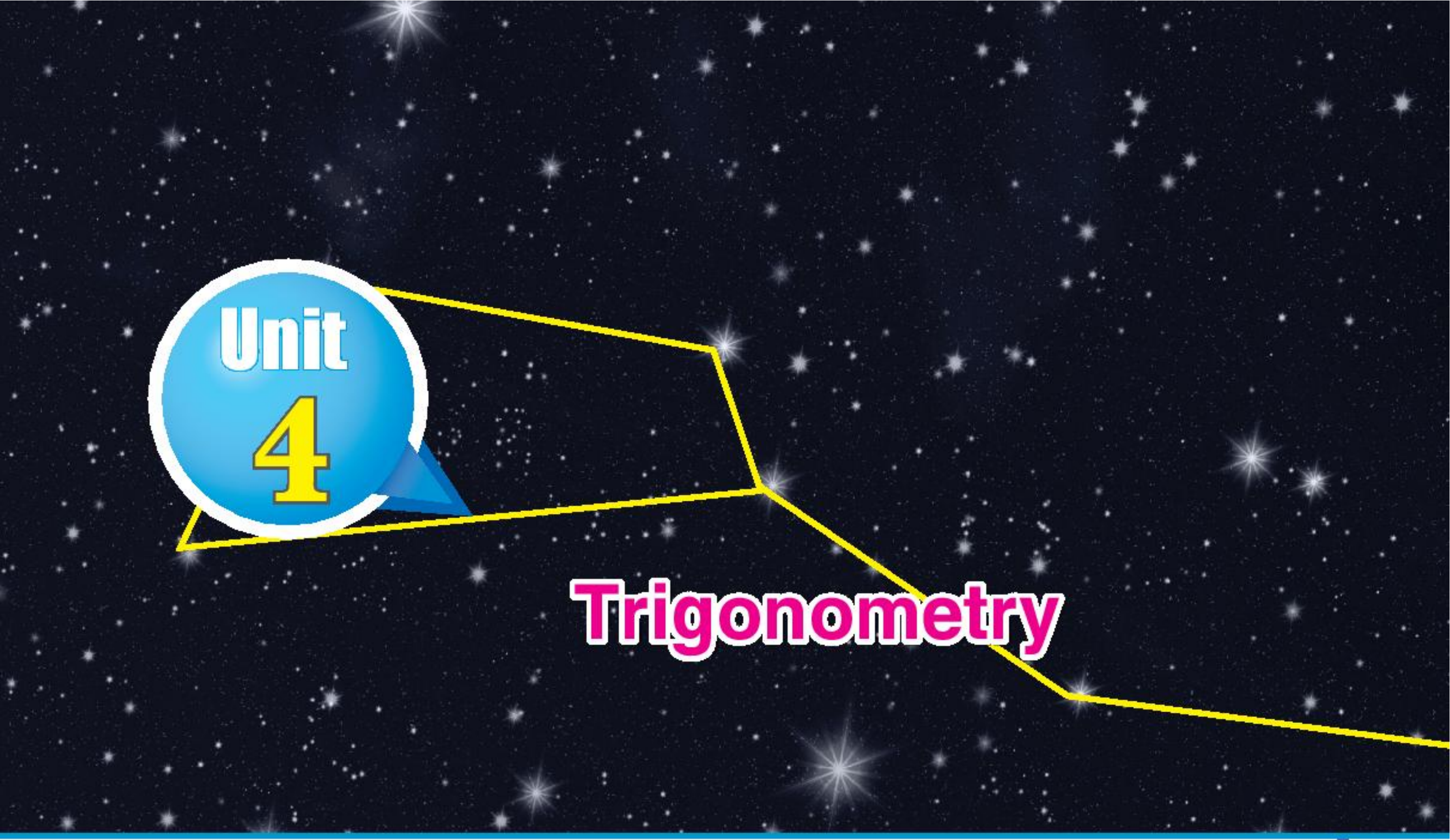
7 In each of the following figures, prove that $\overrightarrow{BE}$ bisects $\angle ABC$.

A



B





Unit 4

Trigonometry

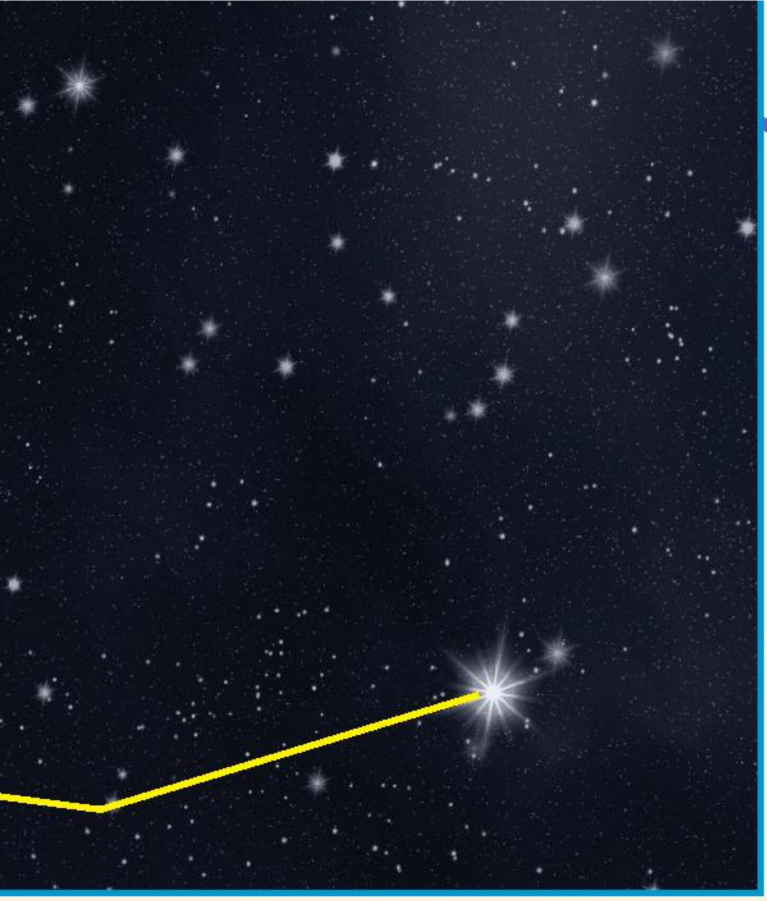
Unit objectives

By the end of the unit, the student should be able to:

- Recognize the directed angle.
- Recognize the standard position of the directed angle .
- Recognize the positive and negative measure of the directed angle.
- Recognize the type of measuring angle (degree and radian measures)
- Recognize the radian measure of the central angles in circle.
- Use the calculator to carry out the special mathematical operation, converting from the radian to the degree measure and vice versa .
- Recognize the triangonometric functions.
- Determine the signs of the trigonometric functions in the four quadrants.
- Deduce the set of equivalent angles which have the same trigonometric functions.
- Recognize the trigonometric ratios of an acute and any angle.
- Deduce the trigonometric ratios to some special angles.
- Recognize the related angles $(180^\circ \pm \theta)$, $(360^\circ \pm \theta)$, $(90^\circ \pm \theta)$, and $(270^\circ \pm \theta)$
- Give the general solution to the trigonometric equations in the form : Sin AX = cos Bx
  tan AX = cot BX   Sec AX = CSC BX
- Find the measure of an angle given one of its trigonometric ratios.
- Recognize the graphic representation to the sine and cosine functions, and deduce the properties of each of them.
- Use the scientific calculator to calculate the trigonometric ratios of some special angles.
- Model some of the physical and life phenomena which are represented by the trigonometric functions.
- Use information technology to recognize the multiple applications of the basic concepts of trigonometry .

Key - Terms

- Degree Measure
- Radian Measure
- Directed Angle
- Radian
- Standard Position
- Positive Measure
- Negative Measure
- Equivalent Angle
- Quadrant Angle
- Trigonometric Function
- Sine
- Cosine
- Tangent
- Cosecant
- Secant
- Cotangent
- Circular Function



Lessons of the unit

Lesson (4 - 1): Directed Angle.
Lesson (4 - 2): Systems of Measuring Angle.
Lesson (4 - 3): Trigonometric Functions.
Lesson (4 - 4): Relations between Trigonometric Functions.
Lesson (4 - 5): Graphing Trigonometric Functions.
Lesson (4 - 6): Finding the Measure of an Angle Given the value of one of its ratios.

Materials

Scientific calculator – Graph calculator – Squared paper – Computer – Graphic programs

Brief History

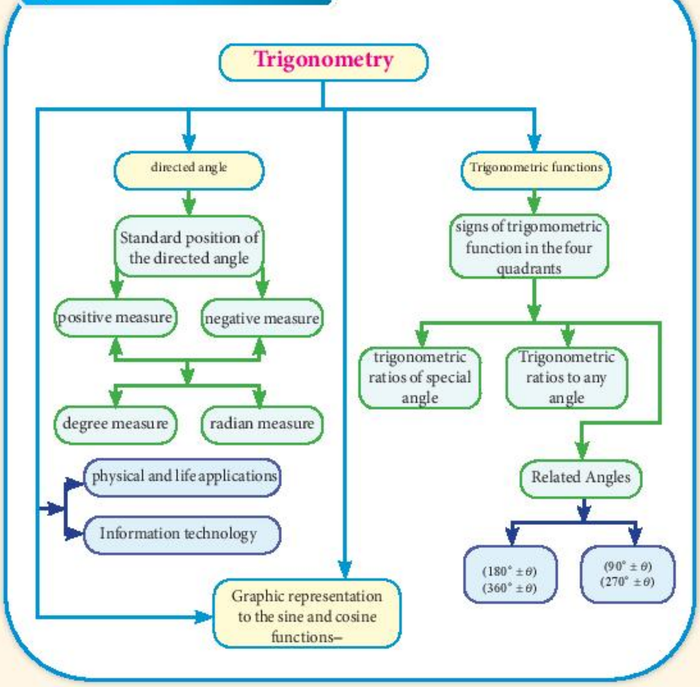
Trigonometry is one of the branches of mathematics. It specializes in calculations among the measures of angles of the triangle and the length of its sides. This science emerged within the ancient mathematics, especially with regard to the calculation of astronomy in which our ancestors were interested in as they watched and contemplated the universe and the movement of the sun, the moon, the stars, and the planets.

The Arab mathematician Nosir-eldin Altousi is the first to separates the trigonometry from the astronomy.

Trigonometry is one of the sciences the arabs were intersted in the arab scientist Abual - Wafa Buzjaty (940-998 AD) in the tenth century describes the terminology " the tangent " and this term is taken from the shadows objects which formed as a result of validity of the emitted light from the sun in a straight lines.

Arabs have many additions in the plane and the circular (w.r. to the sphere) trigonometry, we sternesn we taking from them important information, and they added to it two much until trigonometry became including many mathematical researches, and its applications have become in various scientific knowledge and practical , and also contributed to the advancing the progress and prosperity

Chart of the unit



4 - 1 Directed Angle

You will Learn

- Concept of directed angle.
- Standard position of directed angle.
- Positive and negative measure of the directed angle.
- Position of the directed angle in the coordinates plane.
- Concept of the equivalent angles.

Key - Terms

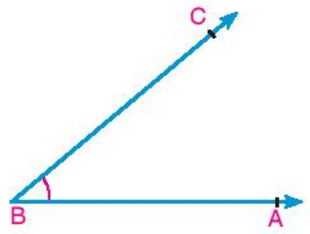
- Degree Measure
- Directed angle
- Standard Position
- Positive measure
- Negative measure
- Equivalent Angle
- Quadrantal Angle

Materials

- Scientific calculator.

Think and discuss

The angle has been defined before as the union of two rays with a common vertex. In the figure opposite, the common point B is called «**Vertex**» of the angle and the two rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are called "**sides**" of the angle.



i.e.: $\overrightarrow{BA} \cup \overrightarrow{BC} = (\angle ABC)$

and is written as $\widehat{ABC}$.

Degree Measure System

You have known that the degree measure depends on dividing the circle into 360 equal arcs in length, then:

1- The central angle subtends one of these arcs, its measure equals one degree (1°)

2- Each degree is subdivided into 60 equal divisions , each division in called a minute and is denoted by ($1'$)

3- Each minute is subdivided into 60 equal divisions, each division is called " a second" and is denoted by ($1''$)

i.e.: $1^\circ = 60'$ and $1' = 60''$

Learn

Directed Angle

We shall now put a further emphasis on the order of the two rays forming the angle, then it is written in the form of an ordered pair ($\overrightarrow{OA}$, $\overrightarrow{OB}$) where $\overrightarrow{OA}$ is the initial side and $\overrightarrow{OB}$ is the terminal side of the angle of vertex O as in figure (1) .

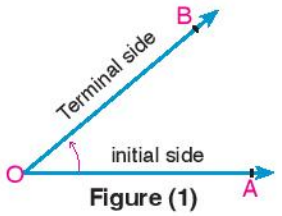


Figure (1)

If the initial side is $\overrightarrow{OB}$ and the terminal side is $\overrightarrow{OA}$, then it is writtern as ($\overrightarrow{OB}$, $\overrightarrow{OA}$) in figure (2)).

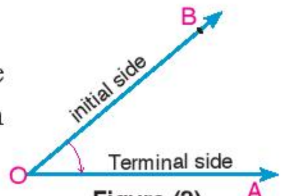


Figure (2)

Definition: the directed angle is an ordered pair of two rays called the sides of the angle with a common starting point called the vertex.

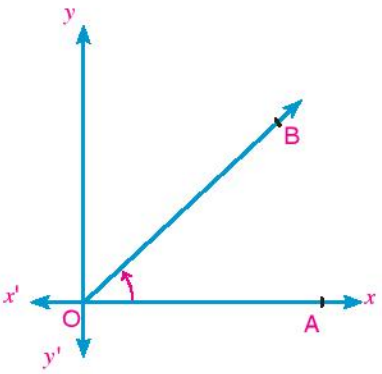
critical thinking:

- Is ($\overrightarrow{OA}$, $\overrightarrow{OB}$) = ($\overrightarrow{OB}$, $\overrightarrow{OA}$)? Explain your answer.

Standard position of the directed angle

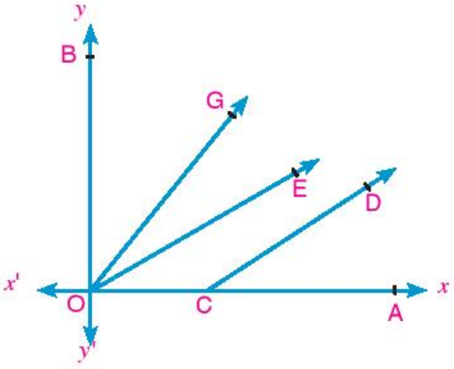
An angle is in the standard position if its vertex is the origin of rectangular coordinate system, and its initial side lies on the positive direction of the x-axis.

Is the directed angle $\angle AOB$ in the standard position? Explain your answer.



Oral exercises

Which one of the following ordered pairs expresses a directed angle in its standard position? Explain your answer .

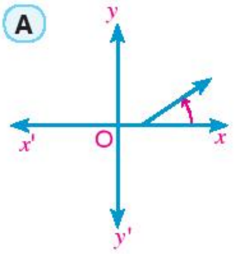


- **A** ($\overrightarrow{CA}$, $\overrightarrow{CD}$)
- **B** ($\overrightarrow{OA}$, $\overrightarrow{OE}$)
- **C** ($\overrightarrow{OE}$, $\overrightarrow{OA}$)
- **D** ($\overrightarrow{OA}$, $\overrightarrow{OG}$)
- **E** ($\overrightarrow{OB}$, $\overrightarrow{OG}$)
- **F** ($\overrightarrow{OA}$, $\overrightarrow{OB}$)

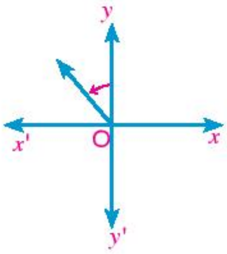
Try to solve

1. Which of the following directed angles is in the standard position? Explain your answer.

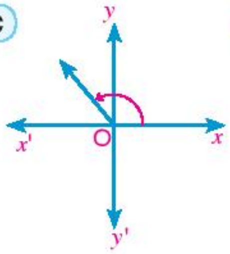
A



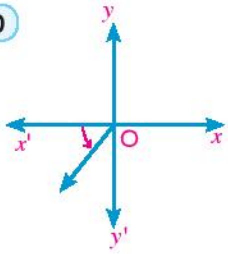
B



C



D



Positive and negative measures of a directed angle

In figure (1) the directed angle, resulting from an anticlockwise rotation has a positive measure.

In figure (2) the directed angle, resulting from a clockwise rotation has a negative measure.

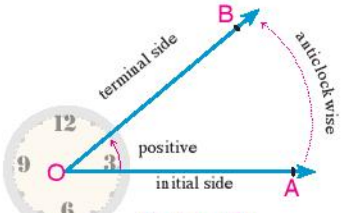


figure (1)

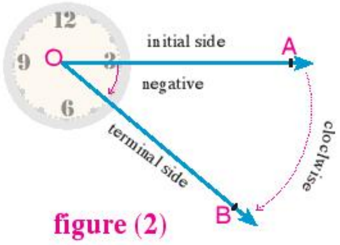
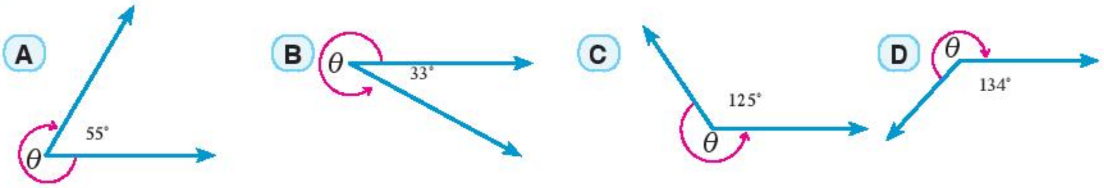


figure (2)

Example

1 Find the measure of the directed angle θ in each of the following figure:



Solution

We know that the sum of measures of accumulative angles around a point equals 360°

A Direction of the angle θ is a clockwise direction $m(\angle\theta) = -(360° - 55°) = -305°$

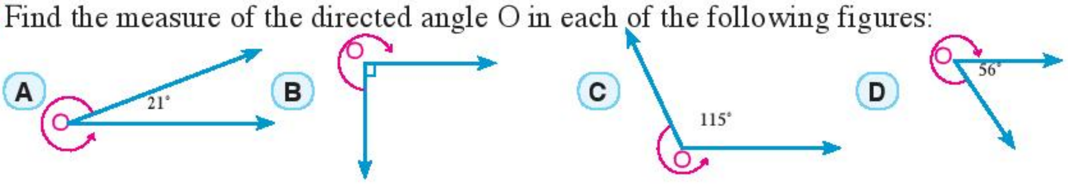
B Direction of the angle θ is an anticlockwise direction $m(\angle\theta) = 360° - 33° = 327°$

C Direction of the angle θ is an anticlockwise direction $m(\angle\theta) = 360° - 125° = 235°$

D Direction of the angle θ is a clockwise direction $m(\angle\theta) = -(360° - 134°) = -226°$

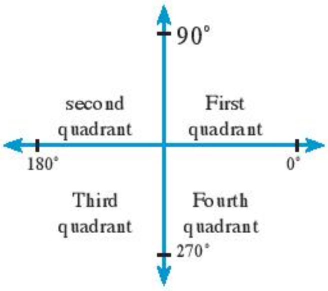
Try to solve

2 Find the measure of the directed angle O in each of the following figures:

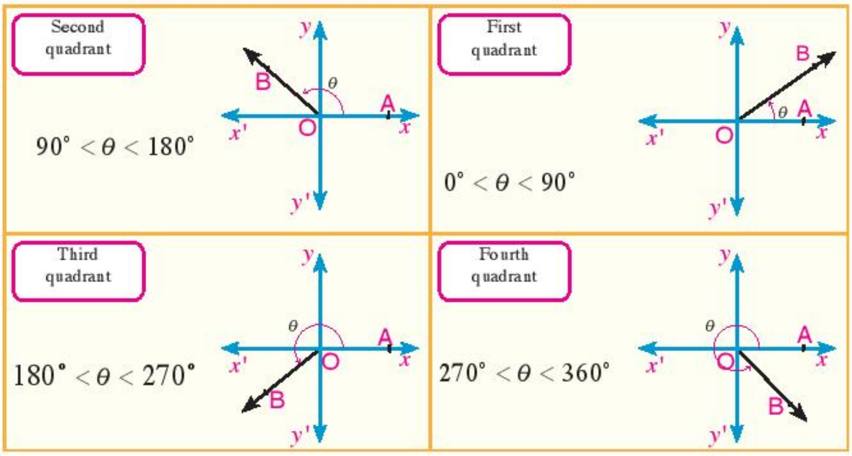


Angle's position in the orthogonal coordinate plane:

➢ The orthogonal coordinate plane is divided into four quadrants as in the figure opposite.



- If the directed angle $\angle$ AOB in the standard position and its positive measure is (θ), then its terminal side $\overrightarrow{OB}$ lies in one of the quadrants:



- If the terminal side $\overrightarrow{OB}$ lies on one of the two axes, then the angle called (Quadrantal angle), and the angles whose measures 0°, 90°, 180°, 270°, 360° are quadrantal angles.

Example

2 Determine the quadrant in which each of the following angles lies:

A 48° **B** 217° **C** 135° **D** 295° **E** 270°

Solution

A $0^\circ < 48^\circ < 90^\circ$ then it lies in the first quadrant.

B $180^\circ < 217^\circ < 270^\circ$ then it lies in the third quadrant.

C $90^\circ < 135^\circ < 180^\circ$ then it lies in the second quadrant.

D $270^\circ < 295^\circ < 360^\circ$ then it lies in the fourth quadrant.

E 270° is a quadrantal angle.

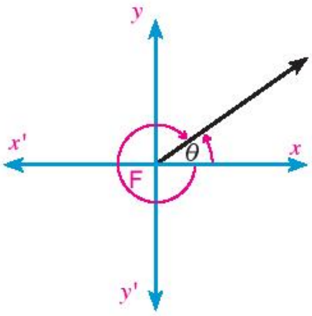
Try to solve

3 Determine the quadrant in which each of the following angles lie:

A 88° **B** 152° **C** 180° **D** 300° **E** 196°

Note:

- If (θ°) is the positive measure of the directed angle, then its negative measure equals $(\theta^\circ - 360^\circ)$
- If ($-\theta^\circ$) in the negative measure of the directed angle, then its positive measure equals $(-\theta^\circ + 360^\circ)$



Example

Add to your knowledge

The sum of the absolute value of each of the positive and negative measure of the directed angle equals 360°

(3) Determine the negative measure of the angle whose measure 275°.

Solution

The negative measure of the angle (275°) = 275° – 360° = – 85°

Check: $|275°| + |-85°| = 275° + 85° = 360°$

Try to solve

(4) Determine the negative measure of the angles whose measures as follows:

(A) 32° (B) 270° (C) 210° (D) 315°

Example

(4) Determine the positive measure of the angle of measure –235°

Solution

The positive measure of the angle (– 235°) = 360° – 235° = 125°

Check: $|-235°| + |125°| = 235° + 125° = 360°$

Try to solve

(5) Determine the positive measure of each of the following angles :

(A) –52° (B) –126° (C) –90° (D) –320°

(6) **Sports:** One of the disc players spins by an angle of measure 150° draw the angle in the standard position.

Equivalent angles

Study the following figures and determine the directed angle (θ) in the standard position in each figure, what do you notice?

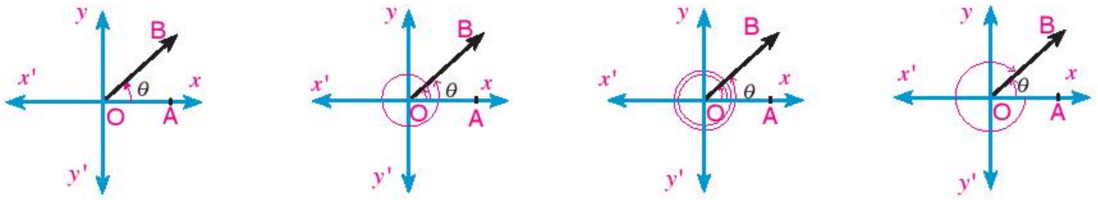


figure (1) figure (2) figure (3) figure (4)

In the figures (1) , (2), (3) and (4), we notice that the angle (θ) and the angle drawn with it have the same side $\overrightarrow{OB}$.

figure (1): the angle of measure θ is in the standard position.

figure (2): the angles θ , $\theta + 360°$ are equivalent.

figure (3): the angles θ , $\theta + 2 \times 360°$ are equivalent.

figure (4): the angles θ , $-(360° - \theta) = \theta - 360°$ are equivalent

From the previous, we deduce that:

When drawing a directed angle θ in the standard position , then all angles whose measures :

$\theta \pm 1 \times 360°$ or $\theta \pm 2 \times 360°$ or $\theta \pm 3 \times 360°$ or.... or $\theta + n \times 360°$ where $n \in Z$

have the same terminal side are called **equivalent angles**.

Example

5 Find a positive and a negative measure of an angle co-terminal with each of the following angles:

A 120°   B – 230°

Solution

A An angle of positive measure: $120° + 360° = 480°$ **(add 360°)**
An angle of negative measure: $120° - 360° = -240°$ **(subtract 360°)**

B An angle of positive measure: $-230° + 360° = 130°$ **(add 360°)**
An angle of negative measure: $-230° - 360° = -590°$ **(subtract 360°)**

Think: Are there other angles of positive measure and others of negative measure? Mention some of these angles if exist.

Try to solve

7 Find a positive and a negative measures of an angle co-terminal with each of the following angles:

A 40°   B 150°   C – 125°   D –240°   E –180°

8 **Discover the error:** all the measures of the following angles are equivalent to the angle of measure 75° in the standard position except:

A –285°   B –645°   C 285°   D 435°

Check your understanding

1 Determine the quadrant in which each of the following angles lies:

A 56°   B 325°   C 570°   D 166°   E 390°

2 Determine a negative measure of each of the following angles of measures:

A 43°   B 214°   C 125°   D 90 °   E 312°

3 Determine the smallest positive measure of each of the following angles:

A –56°   B –215°   C 495°   D 930°   E –450°

Exercises (4 - 1)

1 **Complete:**

A A directed angle is in the standard position if ....

B It is said that the directed angles in the standard position are equivalent if ....

C A directed angle is positive , if the rotation of the angle .... and is negative, if the rotation of the angle ....

D If the terminal side of the directed angle lies on one of the coordinate axes, then it is called ....

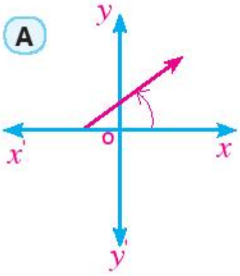
E If (θ) is the measure of a directed angle in the standard position and $n \in Z$, then ($\theta + n \times 360°$) is called .... angles.

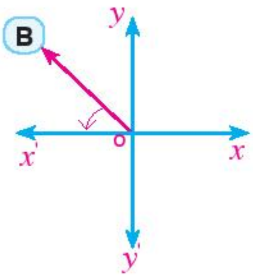
F The smallest positive measure of the angle whose measure 530° is ....

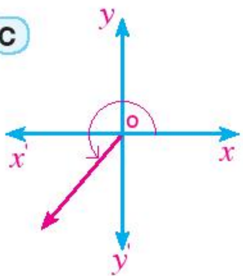
G The angle whose measure 930° lies in the .... quadrant.

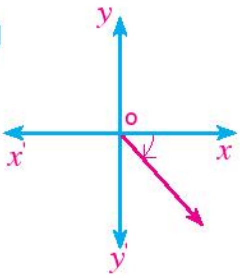
H The smallest positive measure of the angle whose measure –690° is ....

2 Which of the following directed angles is in the standard position

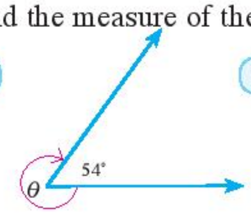
A


B


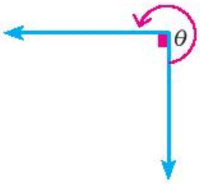
C


D


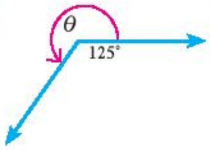
3 Find the measure of the directed angle θ in each of the following figures:

A


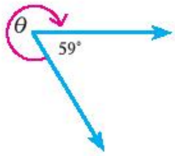
....

B


....

C


....

D


....

4) Determine the quadrant in which each of the following angles lies on:

A) 24° B) 215° C) – 40° D) –220° E) 640°

....

5) Show by drawing each of the following angles in the standard position:

A) 32° B) 140° C) – 80° D) –110° E) –315°

6) Determine a negative measure for each of the following angles:

A) 83° B) 136° C) 90°

....

D) 264° E) 964° F) 1070°

....

7) Determine the smallest positive measure of each of the following angles:

A) –183° B) –217° C) –315° D) –570°

4 - 2 Systems of Measuring Angle

You will Learn

- Concept of radian measure of an angle.
- Relation between radian and degree measure.
- How to find the length of an arc in a circle.

Key - Terms

- Degree Measure
- Radian Measure
- Radian Angle

Materials

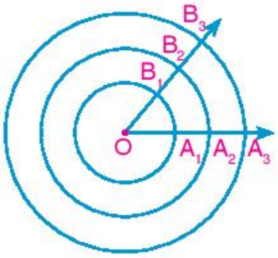
- Scientific calculator.

Think and discuss

You have known that the degree measure is divided into degrees, minutes, and seconds, and one degree = 60 minutes , and one minute = 60 seconds.
Are there other measurements for the angle?

Radian Measure

Group work



1- Draw a set of concentric circles.

2- Find the ratio of the length of the arc and the length of the radius of its corresponding circle – what do you notice?

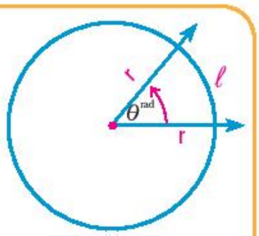
We notice that the ratio of the length of the arc of any central angle, and the radius of its corresponding circle equals constant quantity.

i.e.: $\frac{\text{length of } \widehat{A_1B_1}}{MA_1} = \frac{\text{length of } \widehat{A_2B_2}}{MA_2} = \frac{\text{length of } \widehat{A_3B_3}}{MA_3}$ = constant quantity.

and this constant is the radian measure of the angle. The radian measure of the central angle (θ^{rad}). = $\frac{\text{length of the arc which the central angle subtends}}{\text{Radius of this circle}}$

Definition

If θ^{rad} is the radian measure of the central angle in a circle of radian r subtends an arc of length ℓ, then length of the arc equals the product of the radian measure of the central angle and the radius of its circle:

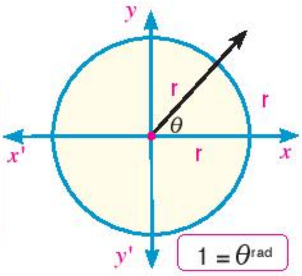


$$\ell = \theta^{rad} \times r$$

From the definition we deduce that:

$$\ell = \theta^{rad} \times r \quad , \quad r = \frac{\ell}{\theta^{rad}}$$

And the unit of measuring angles in the radian measure is the radian angle which is denoted by (1^{rad}) and is read as one radian.



Definition

Radian angle

It is a central angle in a circle subtends an arc of length equals the radius of this circle.

Critical thinking: Is the measure of the central angle in a circle is proportional to the length of the opposite arc? Explain your answer.

Example

(8) A circle of radius 8cm. Find to the nearest hundredth the length of the arc opposite to a central angle of measure $\frac{5}{12}\pi$.

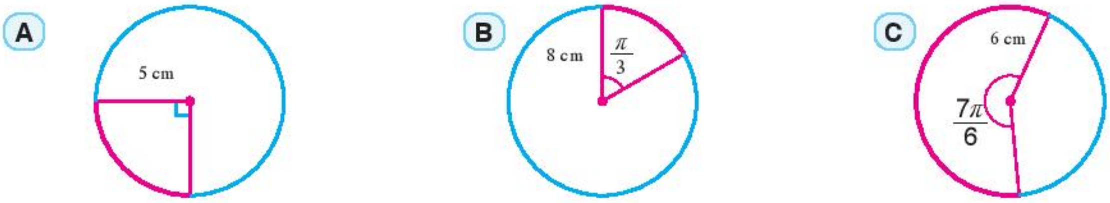
Solution

Use the formula of the length of the arc: $\ell = \theta^{rad} \times r$

$r = 8$ cm , $\theta^{rad} = \frac{5\pi}{12}$, $\ell = \frac{5\pi}{12} \times 8$ $\therefore \ell \simeq 10.47$cm

Try to solve

(1) Find the length of the red arc in each of the following circles approximating the result to the nearest tenth .



Relation between degree measure and radian measure:

You have known that: measure of the central angle in a circle equals the measure of its arc.

i.e. The central angle of degree measure 360°, then the length of its arc equals $2\pi r$

In the unit circle

2π in a radian measure is equivalent to 360° in a degree measure.

i.e. π^{rad} is equivalent to 180° , $1^{rad} = \frac{180^\circ}{\pi} \simeq 57^\circ\ 17'\ 45''$

Add to your knowledge

If the radius of a circle equals the unity then the circle is called the unit circle.

If there is an angle of radian measure θ^{rad} and its degree measure x° then:

$$\frac{x^\circ}{180^\circ} = \frac{\theta^{rad}}{\pi}$$

Example

9. Convert $30°$ to radian measure in terms of π .

To convert to radian measure we use the formula $\frac{x°}{180°} = \frac{\theta^{rad}}{\pi}$

$$\theta^{rad} = \frac{30° \times \pi}{180°} = \frac{\pi}{6}$$

Example

10. Convert 1.2^{rad} to the degree measure.

Solution

$$x° = \frac{1.2 \times 180°}{\pi}$$

$$x° = 68.75493542 = 68°\ 45'\ 18''$$

Calculator is used as follows:

Start → 1 . 2 × 1 8 0 ÷ π = °'''

D MATH
68° 45" 17.77"

Add to your knowlege

There is another unit for measuring angle which is (Grad) and it equals $\frac{1}{200}$ of the measure of the straight angle.
If x, θ, y are three measures of angles in degree, radian, and Grad respectively, then:

$$\frac{x°}{180°} = \frac{\theta^{rad}}{\pi} = \frac{Y^{grad}}{200}$$

Try to solve

2. Convert the measures of the following angles to the degree measure approximating the result to the nearest second:

A) 0.7^{rad} B) 1.6^{rad} C) 2.05^{rad} D) -1.05^{rad}

Exercises (4 - 2)

First: Multiple choice:

1. The angle of measure 60° in the standard position is equivalent to the angle of measure:

   (A) 120° (B) 240° (C) 300° (D) 420°

2. The angle of measure $\frac{31\pi}{6}$ lies in the quadrant

   (A) First (B) second (C) third (D) fourth

3. The angle of measure $\frac{-9\pi}{4}$ lies in the quadrant

   (A) First (B) second (C) third (D) fourth

4. If the sum of measures of the interior angles of a regular polygon equals 180° (n – 2) where n is the number of its sides, then the measure of the angle of a regular pentagon in radian measure equals:

   (A) $\frac{\pi}{3}$ (B) $\frac{7\pi}{2}$ (C) $\frac{3\pi}{5}$ (D) $\frac{2\pi}{3}$

5. The angle of measure $\frac{7\pi}{3}$ its degree measure equals

   (A) 105° (B) 210° (C) 420° (D) 840°

6. If the degree measure of an angle is 64° 48' , then its radian measure equals

   (A) 0.18^{rad} (B) 0.36^{rad} (C) $0.18\,\pi$ (D) $0.36\,\pi$

7. The arc length in a circle of diameter length 24 cm and opposite to a central angle of measure 30° is

   (A) 2π cm (B) 3π cm (C) 4π cm (D) 5π cm

8. The measure of the central angle in a circle of radius length 15 cm and opposite to an arc length 5π cm equals

   (A) 30° (B) 60° (C) 90° (D) 180°

9. If the measure of an angle of a triangle equals 75° and the measure of another angle equals $\frac{\pi}{4}$, then the radian measure of the third angle equals

   (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{5\pi}{12}$

Second: Answer the following questions:

10) In terms of π, find the radian measure of the following angles

A) $225°$ .... B) $240°$ ....

C) $-135°$ .... D) $300°$ ....

E) $390°$ .... F) $780°$ ....

11) Find the radian measure of the following angles approximating the result to the nearest three decimal places:

A) $56.6°$ B) $25°\ 18'$ C) $160°\ 50'\ 48''$

....

12) Find the dgree measure of the following angles approximating the result to the nearest second:

A) 0.49^{rad} B) 2.27^{rad} C) $-3\frac{1}{2}^{rad}$

....

13) θ is a central angle in a circle of radius r and subtends an arc of length L :

A) If $r = 20$ cm and $\theta = 78°\ 15'\ 20''$ then find L. (to the nearest tenth)

B) If $L = 27.3$ cm and $\theta = 78°\ 0'\ 24''$ then find r. (to the nearest tenth)

14) A central angle of measure $150°$ and subtends an arc length 11cm. Calculate its radius length (to the nearest tenth).

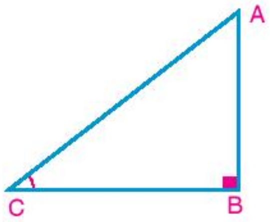
15) Find the radian and degree measure of the central angle which subtends an arc length 8.7cm in a circle of radius length 4cm.

16) **Geometry:** the measure of an angle of a triangle is $60°$ and the measure of another angle is $\frac{\pi}{4}$. Find the radin measure and the degree measure of the third angle.

4 - 3 Trigonometric Functions

Think and discuss

You have studied before the basic trigonometric ratios of an acute angle. In the right angled triangle ABC at B, we get:



$$\sin C = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{AB}{AC}$$

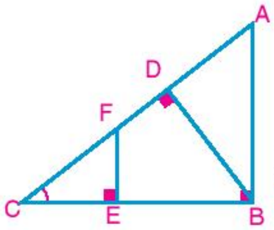
$$\cos C = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{BC}{AC}$$

$$\tan C = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{AB}{BC}$$

1- In the figure opposite, express sin C in three different ratios.

★ Are these ratios equal? Explain your answer.

★ What do you deduce?



Notice that:

The triangles BAC , EFC and DBC are similar (why?)

From similarity , then: $\frac{BA}{AC} = \frac{EF}{FC} = \frac{DB}{BC} = \sin C$ (why?)

i.e.: the trigonometric ratio of an acute angle is constant and does not change except the angle itself is changed.

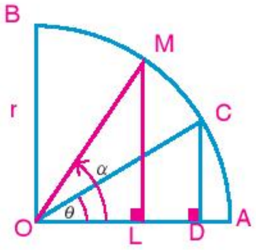
2- The figure opposite shows a quarter of a circle of radius r cm

where: m $(\angle DOC) = \theta$

$$\sin \theta = \frac{CD}{r}$$

when m $(\angle DOC)$ increases to α

then $\sin \alpha = \frac{ML}{r}$



i.e. The trigonometric ratio varies as the measure of its angle , which is known as the trigonometric functions.

You will Learn

- Unit circle.
- Basic trigonometric functions
- Reciprocals of basic trigonometric functions.
- Signs of the trigonometric functions.
- Trigonometric functions of some special angles.

Key - Terms

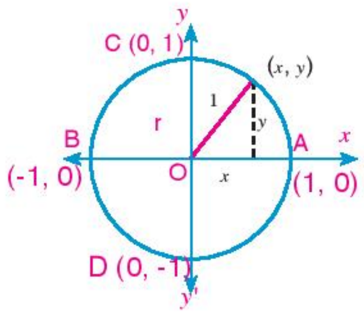
- Trigonometric Function
- Sine (sin)
- Cosine (cos)
- Tangent (tan)
- Cosecant (csc)
- Secant (sec)
- Cotangent (cot)

Learning tools

- Scientific calculator.

The unit circle

In any orthogonal coordinate system, a circle of centre at the origin point and of radius equals the unit of length is called a unit circle.



★ The unit circle intersects the x –axis at the two points A (1, 0) and B (–1, 0), and intersects the y–axis at the two points C (0, 1) and D (0, –1).

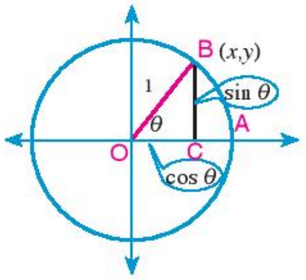
★ If (x, y) are the coordinates of any point on the unit circle:

then $x \in [-1, 1]$, $y \in [-1, 1]$.

where $x^2 + y^2 = 1$ **Phythagorean theorem**

The basic trigonometric functions of an angle

for any directed angle in the standard position, and its terminal side intersects the unit circle at the point B(x, y) and its measure θ , it is possible to define the following functions:



1- cosine of the angle θ = x–coordinate of point B

i.e.: $\cos\theta = x$

2- sine of the angle θ = y– coordinate of point B

i.e.: $\sin\theta = y$

3- tangent of the angle $\theta = \dfrac{y\text{-coordinate of point B}}{x\text{-coordinate of point B}}$

i.e.: $\tan\theta = \dfrac{y}{x}$ where $x \neq 0$, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ where $\cos\theta \neq 0$

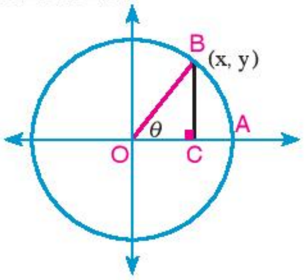
Notice that: the ordered pair (x, y) of any point on the unit circle is written in the form $(\cos\theta, \sin\theta)$

If the point C $\left(\frac{3}{5}, \frac{4}{5}\right)$ is the point of intersection of the terminal side of a directed angle of measure θ with the unit circle

then: $\cos\theta = \frac{3}{5}$, $\sin\theta = \frac{4}{5}$ and $\tan\theta = \frac{4}{3}$

Reciprocals of the basic trigonometric functions

For any directed angle in the standard position and its terminal side intersects the unit circle at the point B(x, y) and its measure is θ , then there are the following functions :



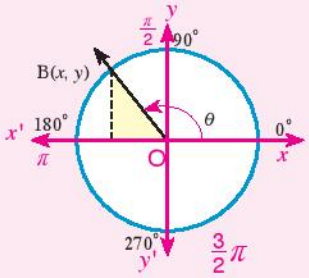
1- secant of the angle θ : $\sec\theta = \dfrac{1}{x} = \dfrac{1}{\cos\theta}$ where $x \neq 0$

2- Cosecant of the angle θ : $\csc\theta = \dfrac{1}{y} = \dfrac{1}{\sin\theta}$ where $y \neq 0$

3- Cotangent of the angle θ : $\cot\theta = \dfrac{x}{y} = \dfrac{1}{\tan\theta}$ where $y \neq 0$

Signs of the Trigonometric Functions

Second quadrant

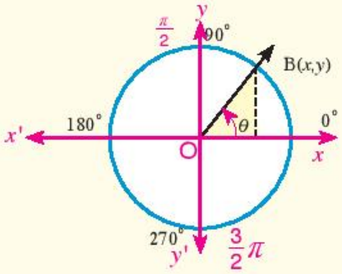


$x < 0$

$y > 0$

The terminal side of the angle lies in the second quadrant,
Thus, the sine and its reciprocal function are positive and the other function are negative.

First quadrant

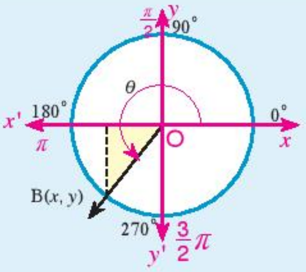


$x > 0$

$y > 0$

The terminal side of the angle lies in the first quadrant.
Thus, all trigonometric functions of the angle whose terminal side $\overrightarrow{OB}$ are positive.

Third quadrant

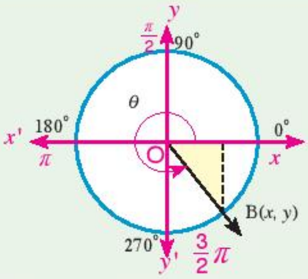


$x < 0$

$y < 0$

The terminal side of the angle lies in the third quadrant.
Thus, the tangent and its reciprocal functions are positive and the other functions are negative.

Fourth quadrant



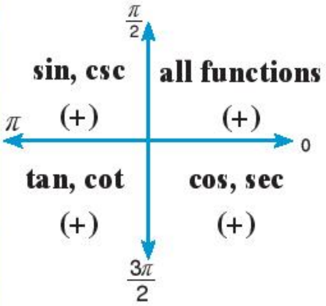
$x > 0$

$y < 0$

The terminal side of the angle lies in the fourth quadrant,
Thus, the cosine and its reciprocal functions are positive and the other functions are negative.

Summary of signs of all trigonometric ratios:

the quadrant in which the terminal side of the angle lies	the interval in which the measure of the angle belongs	signs of trigonometric functions		
		csc , sin	cos , sec	tan , cot
First	$]0 , \frac{\pi}{2}[$	+	+	+
Second	$]\frac{\pi}{2} , \pi[$	+	−	−
Third	$]\pi , \frac{3\pi}{2}[$	−	−	+
Fourth	$]\frac{3\pi}{2} , 2\pi[$	−	+	−



Example

1. Determine the sign of each of the following ratios:

A sin 130°   B tan 315°   C cos 650°   D sec (−30°)

Solution

A The angle of measure 130° lies in the second quadrant   ∴ sin 130° positive

B The angle of measure 315° lies in the fourth quadrant $\therefore$ tan 315° negative

C The angle of measure 650° is equivalent to the angle of measure 650° – 360° = 290°

$\therefore$ The angle of measure 650° lies in the fourth quadrant $\therefore$ cos 650° is positive.

D The angle of measure (–30°) is equivalent to the angle of measure – 30 ° + 360° = 330°

The angle of measure (–30°) lies in the fourth quadrant $\therefore$ sec (–30°) is positive.

Try to solve

1 Determine the sign of each of the following ratios:

A cos 210° **B** sin 740° **C** tan – 300° **D** sin 1230°

Example

2 If $\angle$ AOB is in the standard position and its terminal side intersects the unit circle at the point B and its measure is θ. Find the basic trigonometric ratios to the angle A O B, if the coordinates of the point B are as follows:

A (0, –1) **B** $(\frac{1}{\sqrt{2}}, y)$ **C** $(-x, x)$

where $x > 0$, $y > 0$

Solution

A $\cos\theta = 0$, $\sin\theta = -1$, $\tan\theta = \frac{-1}{0}$ (undefined)

B $x^2 + y^2 = 1$ (unit circle) , $x = \frac{1}{\sqrt{2}}$

$\left(\frac{1}{\sqrt{2}}\right)^2 + y^2 = 1$ then $y^2 = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore y = \frac{1}{\sqrt{2}} > 0$, $y = -\frac{1}{\sqrt{2}} < 0$ (refused)

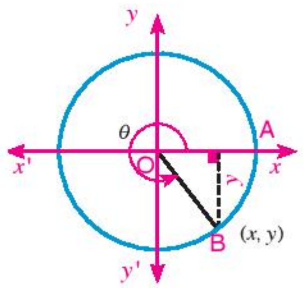
$\therefore \cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{2}}$, $\tan\theta = 1$

C $(-x)^2 + (x)^2 = 1$ $\therefore 2x^2 = 1$ $\therefore x = \frac{1}{\sqrt{2}}$ because $x > 0$

then: $\cos\theta = -\frac{1}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{2}}$, $\tan\theta = -1$

3 If $270^\circ < \theta < 360^\circ$, $\sin\theta = -\frac{5}{13}$ find all basic trigonometric ratios of θ

Solution



let m ($\angle$ AOB) = θ where θ lies in the fourth quadrant , and the coordinates of the point B are (x, y)

$\because y = \sin\theta = -\frac{5}{13}$, $x = \cos\theta$ where $\cos\theta > 0$

$\because x^2 + y^2 = 1$ $\therefore \cos^2\theta + \left(\frac{-5}{13}\right)^2 = 1$

$\therefore \cos^2\theta = 1 - \frac{25}{169}$ $\therefore \cos^2\theta = \frac{144}{169}$, $\cos\theta = \frac{12}{13}$ or $\cos\theta = -\frac{12}{13}$

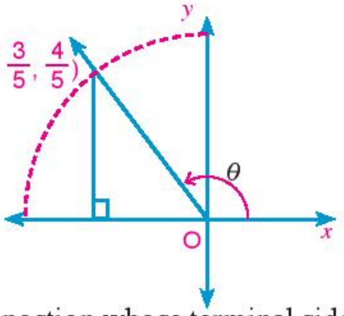
$\cos\theta = \frac{12}{13}$ (why?) $\tan\theta = -\frac{12}{5}$

Try to solve

2) If $90° < \theta < 180°$, $\sin\ \theta = \frac{4}{5}$ find $\cos\theta$, $\tan\theta$ where θ is an angle in the standard position in a unit circle.

Example

4) If the angle θ is drawn in the standard position and its terminal side passes through the point B $(-\frac{3}{5}, \frac{4}{5})$.
then fine all trigonometric ratios of the angle θ.



Solution

$\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{-3}{5} = -\frac{3}{5}$, $\tan\theta = \frac{4}{-3} = -\frac{4}{3}$

$\csc\theta = \frac{5}{4}$, $\sec\theta = \frac{5}{-3} = -\frac{5}{3}$, $\cot\theta = \frac{-3}{4} = -\frac{3}{4}$

Try to solve

3) Find all trigonometric ratios of angle θ drawn in the standard postion whose terminal side passes through the following points:

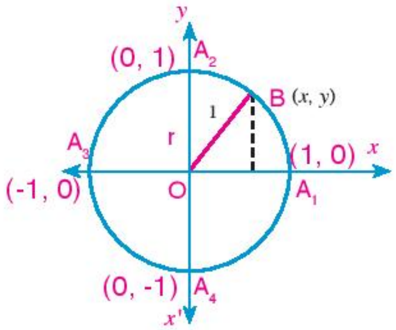
A) $(\frac{5}{13}, \frac{12}{13})$ B) $(\frac{3}{5}, -\frac{4}{5})$ C) $(-\frac{12}{13}, \frac{5}{13})$

Trigonometric ratios of some special angles

In the figure opposite: the unit circle intersected the two axes at the points

$A_1(1, 0)$, $A_2(0, 1)$, $A_3(-1, 0)$, $A_4(0, -1)$.

and θ is the measure of the directed angle A O B in the standard position and its terminal side $\overrightarrow{OB}$ intersects the unit circle at B.



first: If $\theta = 0°$ or $\theta° = 360°$ **at:** B(1, 0)
then: $\cos 0° = \cos 360° = 1$, $\sin 0° = \sin 360° = 0$,
$\tan 0° = \tan 360° = 0$

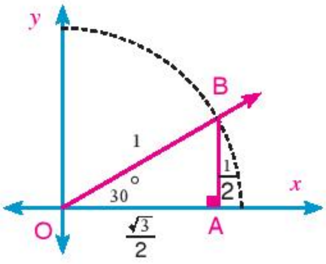
second: If $\theta° = 90° = \frac{\pi}{2}$ **at:** B(0, 1)
then: $\cos 90° = 0$, $\sin 90° = 1$, $\tan = \frac{1}{0}$ (undefied)

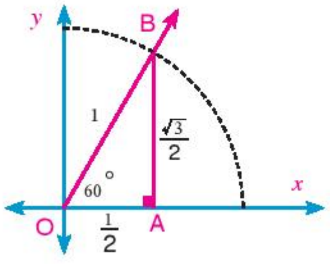
third: If $\theta° = 180° = \pi$ **at:** B(−1, 0)
then: $\cos 180° = -1$, $\sin 180° = 0$, $\tan 180° = 0$

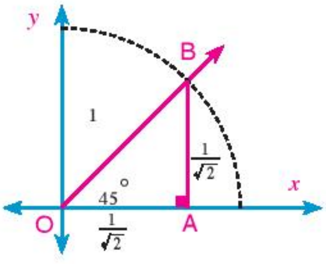
fourth: If $\theta° = 270° = \frac{3\pi}{2}$ **at:** B(0, −1)

then $\cos 270° = 0$, $\sin 270° = -1$, $\tan 270° = \frac{-1}{0}$ (undefined)

In the following figures, determine the coordinates of the point B for each figure and deduce the trigonometric ratios to the measures of angles: 30°, 60°, 45°







Example

5 Prove without using the calculator that: $\sin 60° \cos 30° - \cos 60° \sin 30° = \sin^2 \frac{\pi}{4}$

Solution

You know that $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$, $\sin 60° = \frac{\sqrt{3}}{2}$, $\cos 60° = \frac{1}{2}$

$\therefore$ L.H.S $= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$ **(1)**

$\because \frac{\pi}{4} = 45°$, $\sin 45° = \frac{1}{\sqrt{2}}$

R.H.S $= \sin^2 \frac{\pi}{4} = \sin^2 45° = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$ **(2)**

From (1) and (2) $\therefore$ the two sides are equal.

Try to solve

4 Find the value of: $3 \sin 30° \sin 60° - \cos 0° \sec 60° + \sin 270° \cos^2 45°$

5 **Critical thinking:** If the angle θ is drawn in the standard position and $\cos \theta = \frac{-1}{2}$,

$\sin \theta = \frac{\sqrt{3}}{2}$

Is it possible that $m(\angle \theta) = 240°$? Explain your answer

Check your understanding

Prove that each of the following equality:

A $1 - 2\sin^2 90° = \cos 180°$ **B** $\cos \frac{\pi}{2} = \cos^2 \frac{\pi}{4} - \sin^2 \frac{\pi}{4}$

Exercises (4 - 3)

First: Multiple Choice:

1. If θ is an angle in the standard position and its terminal side passes through the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, then $\sin\theta$ equals:

   A) $\frac{1}{2}$ B) $\frac{1}{\sqrt{3}}$ C) $\frac{\sqrt{3}}{2}$ D) $\frac{2}{\sqrt{3}}$

2. If $\sin\theta = \frac{1}{2}$ where θ is an acute angle , then m ($\angle\theta$) equals

   A) $30°$ B) $45°$ C) $60°$ D) $90°$

3. If $\sin\theta = -1$, $\cos\theta = 0$, then the measure of angle θ equals

   A) $\frac{\pi}{2}$ B) π C) $\frac{3\pi}{2}$ D) 2π

4. If $\csc\theta = 2$ where θ is the measure of an acute angle, then measure of angle θ equals

   A) $15°$ B) $30°$ C) $45°$ D) $60°$

5. If $\cos\theta = \frac{1}{2}$, $\sin\theta = -\frac{\sqrt{3}}{2}$, then measure of angle θ equals

   A) $\frac{2\pi}{3}$ B) $\frac{5\pi}{6}$ C) $\frac{5\pi}{3}$ D) $\frac{11\pi}{6}$

6. If $\tan\theta = 1$ where θ is a positive acute angle, then measure of angle θ equals

   A) $10°$ B) $30°$ C) $45°$ D) $60°$

7. $\tan 45° + \cot 45° - \sec 60°$ equals

   A) Zero B) $\frac{1}{2}$ C) $\frac{\sqrt{3}}{2}$ D) 1

8. If $\cos\theta = \frac{\sqrt{3}}{2}$ where θ is an acute angle , then $\sin\theta$ equals

   A) $\frac{1}{2}$ B) $\frac{1}{\sqrt{3}}$ C) $\frac{2}{\sqrt{3}}$ D) $\frac{\sqrt{3}}{2}$

Second: Answer the following questions:

9. Find all trigonometric functions of angle θ drawn in the standard position and its terminal side intersects the unit circle and passes through each of the following points.

   A) $(\frac{2}{3}, \frac{\sqrt{5}}{3})$ B) $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ C) $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ D) $(-\frac{3}{5}, -\frac{4}{5})$

10. If θ is the measure of the directed angle in the standard position and its terminal side intersects the unit circle at the given point, find all trigonometric function of the angle θ in each of the following cases :

A. $(3a, -4a)$ where $a > 0$

B. $(\frac{3}{2}a, -2a)$ where $\frac{3\pi}{2} < \theta < 2\pi$

11. Determine the sign of each of the following trigonometric function:

A. $\sin 240°$ ....

B. $\tan 365°$ ....

C. $\csc 410°$ ....

D. $\cot \frac{9\pi}{4}$ ....

E. $\sec -\frac{9\pi}{4}$ ....

F. $\tan \frac{-20\pi}{9}$ ....

12. **Find the value of each of the following:**

A. $\cos \frac{\pi}{2} \times \cos 0 + \sin \frac{3\pi}{2} \times \sin \frac{\pi}{2}$

B. $\tan^2 30° + 2 \sin^2 45° + \cos^2 90°$

13. **Discover the error:** The teacher asked the students to find the value of $2 \sin 45°$.

Karim's answer

$2 \sin 45° = \sin 2 \times 45°$
$= \sin 90° = 1$

Ahmed's answer

$2 \sin 45 = 2 \times \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
$= \sqrt{2}$

Which of the two answers is correct? why?

14. **Critical thinking:** If θ is an angle drawn in the standard position where $\cot \theta = -1$ and $\csc \theta = \sqrt{2}$. Is it possible that $m(\angle \theta) = \frac{3\pi}{4}$? Explain your answer.

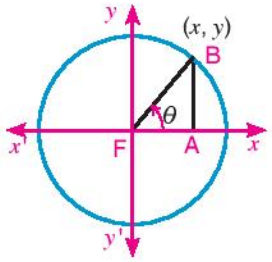
4 - 4

Related Angles

Think and discuss

You have studied before the reflection and you have recognized its properties.

The figure opposite shows the directed angle AOB in the standard position and its terminal side intersects the unit circle at the point B(x, y) and its measure is θ where $0° < \theta < 90°$



Determine the point B' which is the image of the point B by reflection in the y–axis, And mention its coordinates.

What is the measure of ∠ AOB'? Is the angle ∠ AOB' in the standard position?

You will Learn

- Relation between trigonometric functions of angles θ, $180° \pm \theta$
- Relation between trigonometric functions of angles θ, $360° - \theta$
- Relation between trigonometric functions of angles θ, $90° \pm \theta$
- Relation between trigonometric functions of angles θ, $270° \pm \theta$
- The general solution of trigonometric equations in the form:
  - $\sin \alpha = \cos \beta$
  - $\sec \alpha = \csc \beta$
  - $\tan \alpha = \cot \beta$

Key - Terms

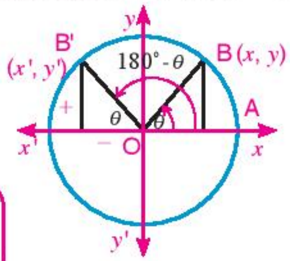
- Related Angles

Learning tools

- Scientific calculator

1- Trigonometric functions of two supplementary angles θ, $(180° - \theta)$

In the figure opposite, B' (x', y') is the image of the point B(x, y) by reflection in the y–axis, then $x' = -x$, $y' = y$ thus:



$\sin (180° - \theta) = \sin \theta$, $\csc (180° - \theta) = \csc \theta$

$\cos (180° - \theta) = -\cos \theta$, $\sec (180° - \theta) = -\sec \theta$

$\tan (180° - \theta) = -\tan \theta$, $\cot (180° - \theta) = -\cot \theta$

For example: $\cos 120° = \cos (180° - 60°) = -\cos 60° = -\frac{1}{2}$

$\sin 135° = \sin (180° - 45°) = \sin 45° = \frac{1}{\sqrt{2}}$

Try to solve

1) find $\tan 135°$, $\sin 120°$, $\cos 150°$

Notice that: $\theta + (180° - \theta) = 180°$

it is said that the two angles θ , $180° - \theta$ **are related angles**.

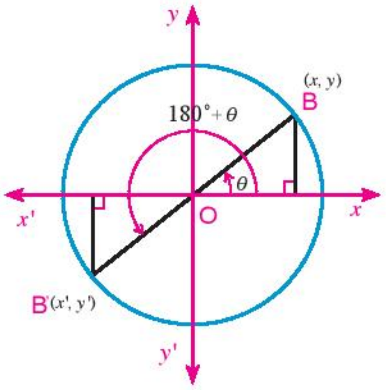
Definition

The related angles: are angles that the difference or the sum of their measures equals a whole number of right angles.

2- Trigonometric functions of angles of measures θ , $(180^\circ + \theta)$

In the figure opposite :

B'(x', y') is the image of the point B(x, y) then $x' = -x$, by reflection in the origin point then $y' = -y$ thus:



$\sin(180^\circ + \theta) = -\sin\theta$,	$\csc(180^\circ + \theta) = -\csc\theta$
$\cos(180^\circ + \theta) = -\cos\theta$,	$\sec(180^\circ + \theta) = -\sec\theta$
$\tan(180^\circ + \theta) = \tan\theta$,	$\cot(180^\circ + \theta) = \cot\theta$

For example:

$\sin 210^\circ = \sin(180^\circ + 30^\circ) = -\sin 30^\circ = -\frac{1}{2}$

$\cos 225^\circ = \cos(180^\circ + 45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$

$\tan 240^\circ = \tan(180^\circ + 60^\circ) = \tan 60^\circ = \sqrt{3}$

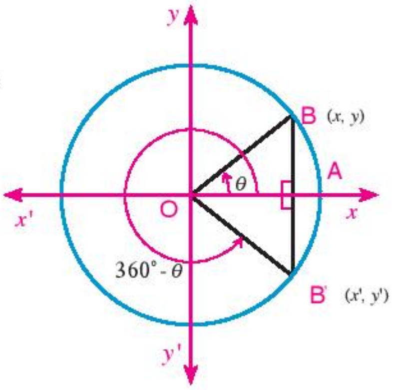
Try to solve

(2) Find sin 225° , cos 210° , sec 600° , cot 225°.

3- Trigonometric functions of angles of measures θ , $(360^\circ - \theta)$

In the figure opposite:

B'(x', y') is the image of the point B(x, y) by reflection in the x-axis then $x' = x$, $y' = -y$:



$\sin(360^\circ - \theta) = -\sin\theta$,	$\csc(360^\circ - \theta) = -\csc\theta$
$\cos(360^\circ - \theta) = \cos\theta$,	$\sec(360^\circ - \theta) = \sec\theta$
$\tan(360^\circ - \theta) = -\tan\theta$,	$\cot(360^\circ - \theta) = -\cot\theta$

For example:

$\sin 330^\circ = \sin(360^\circ - 30^\circ) = -\sin 30^\circ = -\frac{1}{2}$

$\cos 315^\circ = \cos(360^\circ - 45^\circ) = \cos 45^\circ = \frac{1}{\sqrt{2}}$

Try to solve

(3) Find: sin 315° , csc 315° , tan 330° , tan 300°

Critical thinking: How can you find the value of sin (−45°) , cos (−60°) , tan(−30°) , sin 690°.

Notice that

the trigonometric ratios of angle $(-\theta)$ are the same as the trigonometric ratios of angle $(360^\circ - \theta)$

Example

1 Without using the calculator, find the value of the expression :
$\sin 150^\circ \cos(-300^\circ) + \cos 930^\circ \cot 240^\circ$

Solution

$\sin 150^\circ = \sin(180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2}$

$\cos(-300^\circ) = \cos(-300^\circ + 360^\circ) = \cos 60^\circ = \frac{1}{2}$

$\cos 930^\circ = \cos(930^\circ - 2 \times 360^\circ) = \cos 210^\circ$

then $\cos 210^\circ = \cos(180^\circ + 30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$

$\cot 240^\circ = \cot(180^\circ + 60^\circ) = \cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}}$

the expression $= \frac{1}{2} \times \frac{1}{2} + (-\frac{\sqrt{3}}{2}) \times \frac{1}{\sqrt{3}}$

$= \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

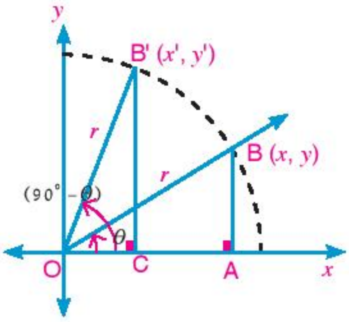
Try to solve

4 Prove that: $\sin 600^\circ \cos(-30^\circ) + \sin 150^\circ \cos(-240^\circ) = -1$

4- Trigonometric functions of two complementary angles θ and $(90^\circ - \theta)$

The figure opposite shows a part of a circle of centre O.
The angle θ is drawn in the standard position to a circle of radius r .
from congruency of the two triangles O A B , O C B':

we get: $x' = y$, $y' = x$



Thus , it is possible to deduce all trigonometric functions of angles θ and $(90^\circ - \theta)$ as follows:

$\sin(90^\circ - \theta) = \cos\theta$, $\csc(90^\circ - \theta) = \sec\theta$

$\cos(90^\circ - \theta) = \sin\theta$, $\sec(90^\circ - \theta) = \csc\theta$

$\tan(90^\circ - \theta) = \cot\theta$, $\cot(90^\circ - \theta) = \tan\theta$

Example

2 If the angle θ is in the standard position, and its terminal side passes through the point $(\frac{3}{5}, \frac{4}{5})$ then find the trigonometric functions: $\sin(90^\circ - \theta)$, $\cot(90^\circ - \theta)$

Solution

$\because \sin(90^\circ - \theta) = \cos\theta$ $\therefore \sin(90^\circ - \theta) = \frac{3}{5}$

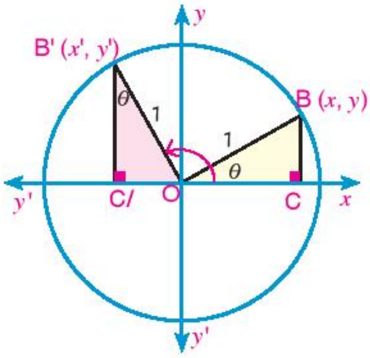
$\because \cot(90^\circ - \theta) = \tan\theta$ $\therefore \cot(90^\circ - \theta) = \frac{4}{3}$

Try to solve

5 In the previous example, find $\cos(90^\circ - \theta)$, $\csc(90^\circ - \theta)$

5- trigonometric functions of angles of measures θ and $(90^\circ + \theta)$

From congruency of the two triangles B' C' O , O C B



We get $y' = x$, $x' = -y$

thus , its is possible to deduce all trigonometric functions of angles θ and $(90^\circ + \theta)$ as follows:

$\sin(90^\circ + \theta) = \cos\theta$,	$\csc(90^\circ + \theta) = \sec\theta$
$\cos(90^\circ + \theta) = -\sin\theta$,	$\sec(90^\circ + \theta) = -\csc\theta$
$\tan(90^\circ + \theta) = -\cot\theta$,	$\cot(90^\circ + \theta) = -\tan\theta$

Example

3 If the angle θ is in the standard position and its terminal side passes through the point $(\frac{1}{3}, \frac{2\sqrt{2}}{3})$

Find the trigonometric functions of : $\tan(90^\circ + \theta)$, $\csc(90^\circ + \theta)$

Solution

$\because \tan(90^\circ + \theta) = -\cot\theta$ $\therefore \tan(90^\circ + \theta) = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}$

$\because \csc(90^\circ + \theta) = \sec\theta$ $\therefore \csc(90^\circ + \theta) = 3$

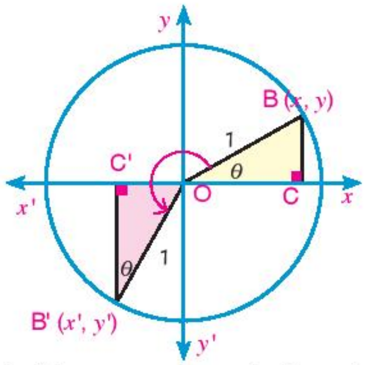
Try to solve

6 In the previous example, find : $\sin(90^\circ + \theta)$, $\sec(90^\circ + \theta)$

6- Trigonometric functions of angles of measures θ and $(270^\circ - \theta)$

From congruency of the two triangles B'C'O , OCB

Thus , it is possible to deduce all trigonometric functions of the two angles θ and $(270^\circ - \theta)$ as follows:



$\sin (270^\circ - \theta) = -\cos \theta$, $\csc (270^\circ - \theta) = -\sec \theta$

$\cos (270^\circ - \theta) = -\sin \theta$, $\sec (270^\circ - \theta) = -\csc \theta$

$\tan (270^\circ - \theta) = \cot \theta$, $\cot (270^\circ - \theta) = \tan \theta$

Example

4) If the angle θ is drawn in the standard position , its terminal side passes through the point $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ then find the trigonometric ratios of : $\cos (270^\circ - \theta)$, $\cot (270^\circ - \theta)$

Solution

$\because \cos (270^\circ - \theta) = -\sin \theta$ $\therefore \cos (270^\circ - \theta) = -\frac{2}{4} = -\frac{1}{2}$

$\because \cot (270^\circ - \theta) = \tan \theta$ $\therefore \cot (270^\circ - \theta) = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$

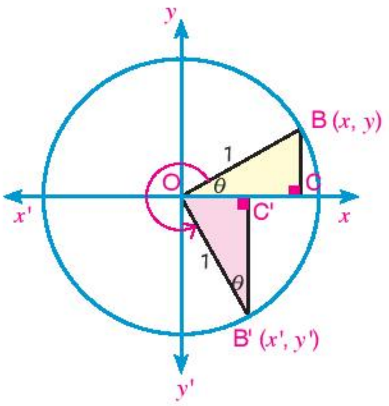
Try to solve

7) In the previous example, find $\tan (270^\circ - \theta)$, $\csc (270^\circ - \theta)$

7- Trigonometric functions of angles of measures θ and $(270^\circ + \theta)$

From congruency of the two triangles: B' C' O, O C B

Thus, it is possible to deduce all trigonometric functions of angles θ and $(270^\circ + \theta)$ as follows:



$\sin (270^\circ + \theta) = -\cos \theta$, $\csc (270^\circ + \theta) = -\sec \theta$

$\cos (270^\circ + \theta) = \sin \theta$, $\sec (270^\circ + \theta) = \csc \theta$

$\tan (270^\circ + \theta) = -\cot \theta$, $\cot (270^\circ + \theta) = -\tan \theta$

Example

5) If the angle θ is in the standard position , its terminal side passes through the point $(\frac{\sqrt{5}}{3}, \frac{2}{3})$ then find the trigonometric ratios of: $\sin (270^\circ + \theta)$, $\sec (270^\circ + \theta)$

Solution

$\because \sin(270° + \theta) = -\cos\theta$ $\therefore \sin(270° + \theta) = -\frac{\sqrt{5}}{3}$

$\because \sec(270° + \theta) = \csc\theta$ $\therefore \sec(270° + \theta) = \frac{3}{2}$

Try to solve

8) In the previous example, find $\cot(270° + \theta)$, $\csc(270° + \theta)$.

General solution of trigonometric equations in the form

[sin(α) = cos(β), sec(α) = csc (β), tan(α) = cot(β)]

Think and discuss

you have studied before that, if α and β are the measures of two complementary angles (their sum equals 90°) then $\sin\alpha = \cos\beta$, $\sec\alpha = \csc\beta$, $\tan\alpha = \cot\beta$, hence $\alpha + \beta = 90°$ where α and β are two acute angles , If $\sin\theta = \cos 15°$ then what are expected values of θ?

Learn

1- If $\sin\alpha = \cos\beta$ (where α, β are the measures of two complementary angles) then:

- $\sin\alpha = \sin(\frac{\pi}{2} - \beta)$ **hence:** $\alpha = \frac{\pi}{2} - \beta$ **i.e.** $\alpha + \beta = \frac{\pi}{2}$
- $\sin\alpha = \sin(\frac{\pi}{2} + \beta)$ **hence:** $\alpha = \frac{\pi}{2} + \beta$ **i.e.** $\alpha - \beta = \frac{\pi}{2}$

Add 2πn (where n $\in$ Z) to the angle $\frac{\pi}{2}$ then:

when $\sin\alpha = \cos\beta$ **then** $\alpha \pm \beta = \frac{\pi}{2} + 2\pi n$

(n $\in$ Z), similarly:

when $\csc\alpha = \sec\beta$ **then** $\alpha \pm \beta = \frac{\pi}{2} + 2\pi n$

(n $\in$ Z),
$\alpha \neq n\pi$, $\beta \neq (2n + 1)\frac{\pi}{2}$

2- If $\tan\alpha = \cot\beta$ (where α, β are the measure of two complementary angles) then :

- $\tan\alpha = \tan(\frac{\pi}{2} - \beta)$ **hence:** $\alpha = \frac{\pi}{2} - \beta$ **i.e.** $\alpha + \beta = \frac{\pi}{2}$
- $\tan\alpha = \tan(\frac{3\pi}{2} - \beta)$ **hence:** $\alpha = \frac{3\pi}{2} - \beta$ **i.e.** $\alpha + \beta = \frac{3\pi}{2}$

Add 2πn (where n $\in$ Z) to the two angles $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ then :

when $\tan\alpha = \cot\beta$ **then** $\alpha + \beta = \frac{\pi}{2} + \pi n$

(where n $\in$ Z),
$\alpha \neq (2n + 1)\frac{\pi}{2}$, $\beta \neq n\pi$

Example

6 Solve the equation: $\sin 2\theta = \cos\theta$

Solution

The equation: $\sin 2\theta = \cos\theta$

$2\theta \pm \theta = \frac{\pi}{2} + 2\pi n \quad (n \in Z)$ **from definition of equation**

(1) **either** $2\theta + \theta = \frac{\pi}{2} + 2\pi n$ **i.e:** $3\theta = \frac{\pi}{2} + 2\pi n$

$\theta = \frac{\pi}{6} + \frac{2}{3}\pi n$ **divide both sides by 3**

(2) **or** $2\theta - \theta = \frac{\pi}{2} + 2\pi n$ **i.e.:** $\theta = \frac{\pi}{2} + 2\pi n$

Solution of the equation: $\frac{\pi}{6} + \frac{2}{3}\pi n$ or $\frac{\pi}{2} + 2\pi n$

Try to solve

9 Find the general solution of each of the following equations:

A $\sin 4\theta = \cos 2\theta$ B $2\sin(\frac{\pi}{2} - \theta) = 1$ C $\cos 5\theta = \sin\theta$

10 **Discover the error:** In one of the mathematical competitions, the teacher asked karim and Ziad to find the value of $\sin(\theta - \frac{\pi}{2})$ then who of them has a correct answer? Explain your answer.

karim's answer

$\sin(\theta - \frac{\pi}{2}) = \sin(2\pi + \theta - \frac{\pi}{2})$

$= \sin(\frac{3}{2}\pi + \theta)$

$= -\cos\theta$

Ziad's answer

$\sin(\theta - \frac{\pi}{2}) = \sin[-(\frac{\pi}{2} - \theta)]$

$= -\sin(\frac{\pi}{2} - \theta)$

$= -(-\cos\theta) = \cos\theta$

Check your understanding

Find all values of θ where $\theta \in]0, \frac{\pi}{2}[$ which satisfies each of the following equalities :

A $\sin\theta - \cos\theta = 0$ B $\csc(\theta - \frac{\pi}{6}) = \sec\theta$ C $2\cos(\frac{\pi}{2} - \theta) = 1$

Exercises (4 - 4)

First: Complete each of the following:

1. $\cos(180° + \theta) =$ ....
2. $\tan(180° - \theta) =$ ....
3. $\csc(360° - \theta) =$ ....
4. $\sin(360° + \theta) =$ ....
5. $\sin(90° + \theta) =$ ....
6. $\cot(90° - \theta) =$ ....
7. $\sec(270° + \theta) =$ ....
8. $\cos(270° - \theta) =$ ....

Second: Complete each of the following with a measure of an acute angle

9. $\sin 25° = \cos$ ....°
10. $\cos 67° = \sin$ ....°
11. $\tan 42° = \cot$ ....°
12. $\csc 13° = \sec$ ....°
13. If $\cot an\ 2\theta = \tan\theta$ where $0° < \theta < 90°$ then $m(\angle \theta) =$ ....
14. If $\sin 5\theta = \cos 4\theta$ where θ is a positive acute angle , then $\theta =$ ....°
15. If $\sec\theta = \sec(90° - \theta)$, then $\cot\theta =$ ....
16. If $\tan 2\theta = \cot 3\theta$ where $\theta \in]0, \frac{\pi}{2}[$, then $m(\angle \theta) =$ .... rad
17. If $\cos\theta = \sin 2\theta$ where θ is a positive acute angle, then $\sin 3\theta =$ ....

Third: Multiple choice:

18. If $\tan(180° + \theta) = 1$ where θ is the measure of the smallest positive angle , then measure of θ equals

   A) $45°$ B) $30°$ C) $60°$ D) $135°$

19. If $\cos 2\theta = \sin\theta$ where $\theta \in]0, \frac{\pi}{2}[$ then $\cos 2\theta$ equals

   A) $\frac{1}{\sqrt{2}}$ B) $\frac{1}{2}$ C) $\frac{\sqrt{3}}{2}$ D) 1

20. If $\sin\alpha = \cos\beta$ where α and β are two acute angles, then $\tan(\alpha + \beta)$ equals

   A) $\frac{1}{\sqrt{3}}$ B) 1 C) $\sqrt{3}$ D) undefined

21. If $\sin 2\theta = \cos 4\theta$ where θ is a positive acute angle, then $\tan(90° - 3\theta)$ equals

   A) -1 B) $\frac{1}{\sqrt{3}}$ C) 1 D) $\sqrt{3}$

22 If $\cos(90^\circ + \theta) = \frac{1}{2}$ where θ is the measure of the smallest positive angle , then measure of angle θ equals

A 150° B 210° C 240° D 330°

Fourth: Answer the following question:

23 Find one of the values of θ where $0 \leqslant \theta < 90^\circ$ which satisfies each of the following:

A $\sin(3\theta + 15^\circ) = \cos(2\theta - 5^\circ)$

B $\sec(\theta + 25^\circ) = \csc(\theta + 15^\circ)$

C $\tan(\theta + 20^\circ) = \cot(3\theta + 30^\circ)$

D $\cos \frac{\theta + 20^\circ}{2} = \sin \frac{\theta + 40^\circ}{2}$

24 Find the value of each of the following:

A $\sin 150^\circ$ B $\csc 225^\circ$ C $\sec 300^\circ$ D $\tan 780^\circ$

....

E $\csc \frac{11\pi}{6}$ F $\sin \frac{7\pi}{4}$ H $\cot \frac{-2\pi}{3}$ I $\cos \frac{-7\pi}{4}$

....

25 If the terminal side of the angle θ drawn in the standard position intersects the unit circle at the point B $(-\frac{3}{5}, \frac{4}{5})$, then find:

A $\sin(180^\circ + \theta)$

....

B $\cos(\frac{\pi}{2} - \theta)$

....

C $\tan(360^\circ - \theta)$

....

D $\csc(\frac{3\pi}{2} - \theta)$

....

4 - 5 Graphing Trigonometric Functions

You will Learn

- Graph the sine function , and deduce its properties.
- Graph the cosine function, and deduce its properties.

Key - Terms

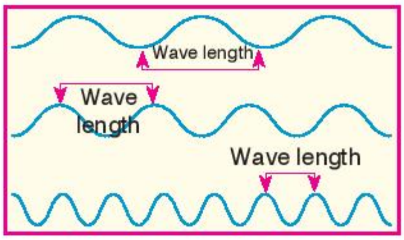
- Sine Function
- Cosine Function
- Maximum Value
- Minimum Value

Learning tools

- Graphic calculator .
- Computer
- Graphic programs.

Think and discuss

Ultrasound depend on high frequencies, differ in wave length, as used in medical photography and submarines use it as a radar works in the depths of the ocean. When these waves are represented Graphically to know the properties of the sine function and cosine function, work in group with your classmate the following:



Graphical representation of the sine function

Group work

1 Complete the following table:

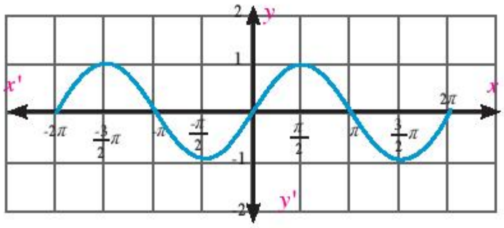
θ	0	$\frac{\pi}{6}$	$\frac{3\pi}{6}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{9\pi}{6}$	$\frac{11\pi}{6}$	2π
$\sin\theta$	0	0,5							

2 Draw the curve by connecting its all points.

3 Construct another table uses the additive inverse of the given values in the previous table.

4 Determine all points which you have got on the coordinates lattice.

5 Complete drawing the curve by connecting all its points.



6 Are you notice the existence of maximum or minimum values to this curve? explain your answer.

Learn

Properties of the sine function

In the function f where $f(\theta) = \sin\theta$ then:

★ The domain of the sine function is $]-\infty, \infty[$, and its range $[-1, 1]$

★ The sine function is periodic with period 2π i.e. it is possible to shift the curve in the interval $[0, 2\pi]$ to the right or to the left 2π unit, 4π unit, 6π unit, ... and so on.

★ The maximum value of the sine function equals 1 and takes place at the points
$\theta = \frac{\pi}{2} + 2n\pi \qquad n \in z$

★ The minimum value of the sine function equals -1 and takes place at the points
$\theta = \frac{3\pi}{2} + 2n\pi \qquad n \in z$

Graphical representation of the cosine function

Group work

1 Complete the following table with your classmate:

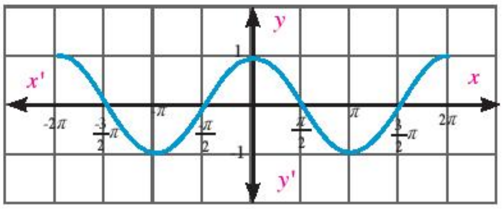
θ	0	$\frac{\pi}{6}$	$\frac{3\pi}{6}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{9\pi}{6}$	$\frac{11\pi}{6}$	2π
$\cos\theta$	1	0,8							

2 Draw the curve by connecting all its points.

3 Construct another table uses the additive inverse of the given values in the previous table.

4 Determine all points which you have got on the coordinates lattice.

5 Complete drawing the curve by connecting all its points.



Learn

Properties of the cosine function

In the function f where $f(\theta) = \cos\theta$ then:

★ The domain of the cosine function is $]-\infty, \infty[$, and its range is $[-1, 1]$

★ The cosine function is periodic with period 2π, it is possible to shift the curve in the interval $[0, 2\pi]$ to the right or to the left 2π unit, 4π unit , 6π unit , ... and so on.

★ The maximum value of the cosine function equals 1 and takes place at the points $\theta = \pm 2 n\pi$ $\quad n \in Z$

★ The minimum value of the sine function equals – 1 and takes place at the points $\theta = \pi \pm 2n\pi$ $\quad n \in Z$

Example

1 **Physics:** It is possible to the ships entering the port, if the level of water is high as a result of the movement of the ebb and tide , where the depth of water is at least 10 metres. The movement of ebb and tide in that day is given by the relation, $S = 6 \sin (15\,n)^\circ + 10$ where n is the time elapsed after the mid-night in hours according to (24 hours system). How many times did the depth of water completely reach 10 metres in the port. Draw a graph representation to show how the depth of water vary with the movement of ebb and tide during the day.

Solution

The relation between the time (n) in hours and the depth of water (s) in metres

from the relation: $S = 6 \sin (15\, n)^\circ + 10$

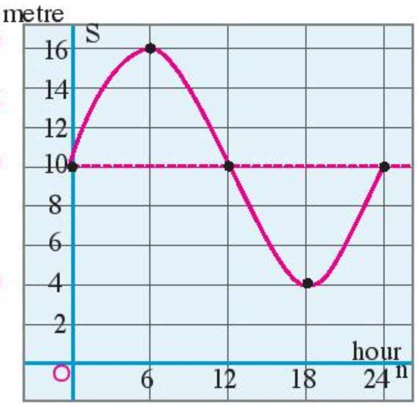
when n = 0 $S = 6 \sin (15 \times 0) + 10 \quad = 6 \sin 0 + 10 \quad = 10$

when n = 6 $S = 6 \sin (15 \times 6) + 10 \quad = 6 \sin 90^\circ + 10 \quad = 16$

when n = 12 $S = 6 \sin (15 \times 12) + 10 = 6 \sin 180^\circ + 10 = 10$

when n = 18 $S = 6 \sin (15 \times 18) + 10 = 6 \sin 270^\circ + 10 = 4$

when n = 24 $S = 6 \sin (15 \times 24) + 10 = 6 \sin 360^\circ + 10 = 10$



n in hours	0	6	12	18	24
S in metres	10	16	10	4	10

From the table we get : the depth of water reaches 10 metres when n = 10, 12, 24 hours

Try to solve

1 In the previous example , How many hours during the day at which the ship can able to enter the port?

Check your understanding

1 Draw the graph of the function $y = 3 \sin x$ where $x \in [0, 2\pi]$

2 Draw the graph of the function $y = 2\cos x$ where $x \in [0, 2\pi]$

Exercises (4 - 5)

First: complete each of the following:

1. The range of the function f where $f(\theta) = \sin\theta$ is ....
2. The range of the function f where $f(\theta) = 2\sin\theta$ is ....
3. The maximum value of the function f where $f(\theta) = 4\sin\theta$ is ....
4. The minimum value of the function f where $f(\theta) = 3\cos\theta$ is ....

Second: write the rule for each trigonometric function beside the corresponding figure to it.

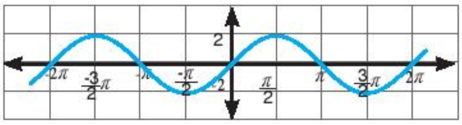


Figure (1) the rule is:

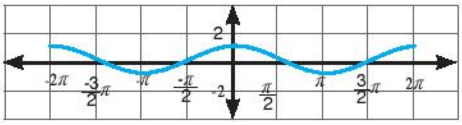


Figure (2) the rule is:

Third: Answer the following questions:

5. Find the maximum and minimum values, then calculate the range of each o the following functions :

   A. $y = \sin\theta$

   B. $y = 3\cos\theta$

   C. $y = \frac{3}{2}\sin\theta$

4 - 6

Finding the Measure of an Angle Given the Value of One of its Trigonometric Ratios

You will Learn

- Find the measure of an angle given a trigonometric function.

Key - Terms

- Trigonometric Function

Learning tools

- Scientific calcualtor

Think and discuss

You have known that , if $y = \sin\theta$, then it is possible to find the value of y given the angle θ, and when the value of y is given then, is it possible to find the value of θ ?

Learn

If $y = \sin\theta$ **then** $\theta = \sin^{-1} y$

For example if θ is a positive acute angle and $y = \frac{1}{2}$,

then this relation can be written in the form $\theta = \sin^{-1}\frac{1}{2} = 30°$

Example

1. Find m ($\angle\theta$) where $0° < \theta < 360°$ which satisfies each of the following:

   A) $\sin\theta = 0.6325$ B) $\cot\theta = (-1.6204)$

Solution

A) $\because$ Sine of the function > 0

$\therefore$ The angle lies in the first or in the second quadrant .

Use the calculator:

Start → SHIFT sin⁻¹ 0 . 6 3 2 5 = °'''

The first quadrant: $m(\angle\theta) = 39°\ 14'\ 6''$

The Second quadrant: $m(\angle\theta) = 180° - 39°\ 14'\ 6' = 140°\ 45'\ 54''$

B) $\because$ The co-tangent of the angle < 0

$\therefore$ the angle lies in the second or in the fourth quadrant:

Use the calculator:

Start → SHIFT tan⁻¹ 1 . 6 2 0 4 x⁻¹ = °'''

The second quadrant: $m(\angle\theta) = 180° - 31°\ 40'\ 48'' = 148°\ 19'\ 12''$

The fourth quadrant: $m(\angle\theta) = 360° - 31°\ 40'\ 48' = 328°\ 19'\ 12'$

Is it possible to check the answer using the calculator

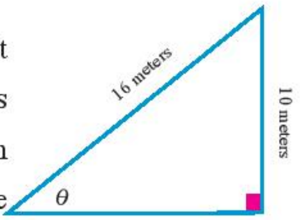
Try to solve

1. Find m($\angle\theta$) where $0° < \theta < 360°$ which satisfies each of the following:

   A) $\cos\theta = 0.6205$ B) $\tan\theta = (-2.3615)$ C) $\csc\theta = (-2.1036)$

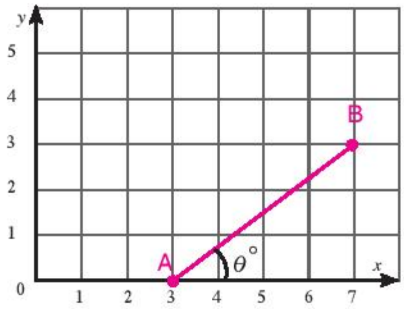
Check your understanding

1. **Sports:** there is a skiing game in the theme parks, if the height of one of these games is 10 metres and its length is 16 metres as in the figure opposite . write a trigonometric function you can use to find the value of the angle θ , then find the value of the angle in degree to the nearest thousandth.



2. **Cars:** Karim descends by his car down a ramp of length 65m and its height is 8 metres. If the ramp makes an angle θ with the horizontal . find m($\angle\theta$) in degree measure.

3. **Critical thinking:** the figure opposite represent a line segment joining between the two points A(3, 0), B (7, 3), find the measure of the angle θ including between $\overline{AB}$ and the x axis .



Exercises (4 - 6)

First : Multiple choice:

1. If $\sin \theta = 0.4325$ where θ is a positive acute angle, then $m(\angle\theta)$ equals
   (A) 25.626° (B) 64.347° (C) 32.388° (D) 46.316°

2. If $\tan \theta = 1.8$ and $90° \leqslant \theta \leqslant 360°$, then $m(\angle\theta)$ equals
   (A) 60.945° (B) 119.055° (C) 240.945° (D) 299.055°

Second : Answer the following questions:

1. If the terminal side of angle θ in the standard position intersects the unit circle at point B, then find each of $\sin\theta$ and $\cos\theta$ in the following cases:
   (A) $B(\frac{1}{2}, \frac{\sqrt{3}}{2})$ (B) $B(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ (C) $B(-\frac{6}{10}, \frac{8}{10})$

   ....

   ....

2. If the terminal side of angle θ in the standard position intersects the unit circle at point B then find each of $\sec\theta$ and $\csc\theta$ in the following cases:
   (A) $B(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ (B) $B(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$ (C) $B(-\frac{5}{13}, -\frac{12}{13})$

   ....

   ....

3. If the terminal side of angle θ in the standard position intersects the unit circle at point B, then find each of $\tan\theta$ and $\cot\theta$ in the following cases::
   (A) $B(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}})$ (B) $B(\frac{3}{\sqrt{34}}, -\frac{5}{\sqrt{34}})$ (C) $B(-\frac{4}{5}, -\frac{3}{5})$

   ....

   ....

4. If the terminal side of angle θ in the standard position intersects the unit circle at point B, then find $m(\angle\theta)$ where $0° < \theta < 360°$ when:
   (A) $B(\frac{\sqrt{3}}{2}, \frac{1}{2})$ (B) $B(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ (C) $B(\frac{6}{10}, \frac{-8}{10})$

   ....

5 Use the degree measure to find the smallest positive angle which satisfies each of the following:

A $\sin^{-1} 0.6$ **B** $\cos^{-1} 0.436$ **C** $\tan^{-1} 1.4552$

D $\sec^{-1} (-2.2364)$ **E** $\cot^{-1} 3.6218$ **F** $\csc^{-1} (-1.6004)$

6 If $0° \leqslant \theta \leqslant 360°$, then find the measure of angle θ in each of the following:

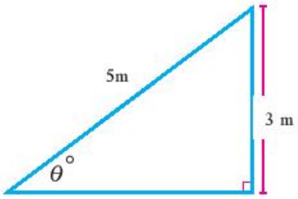
A $\sin^{-1} (0.2356)$ **B** $\cos^{-1} (-0.642)$ **C** $\tan^{-1} (-2.1456)$

7 If $\sin \theta = \frac{1}{3}$ and $90° \leqslant \theta \leqslant 180°$.

A Calculate the measure of angle θ to the nearest second

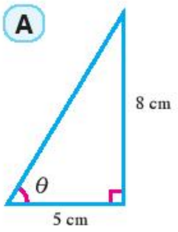
B Find the value of $\cos \theta$, $\tan\theta$ and $\sec\theta$.

8 **Ladder:** A ladder of length 5 metres rests on a wall , if the height of the ladder from the ground is 3 metres. Find in radian the measure of the angle of inclination of the ladder to the horizontal.

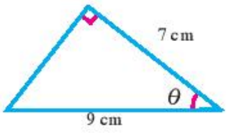
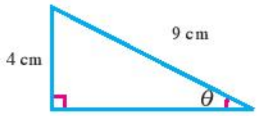


9 Find the degree measure of angle θ in each of the following figures:

A



B



C